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Theory of Jacobi Forms of Degree Two over Cayley Numbers

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ABSTRACT

In this paper, we outline the recent development of the theory of Jacobi forms over Cayley numbers, initiated by the author in 1991, and the theory of Jacobi forms of degree two over Cayley numbers was developed. A family of Eisenstein series is constructed via a group representation derived from the transformation formula of a family of theta series. This provides another example of Jacobi forms besides natural examples from Fourier-Jacobi expansions of modular forms on the exceptional domain of 27 dimensions.

Key Words: Jacobi form, Cayley number, exceptional domain, modular form, theta series

I. Introduction

The theory of Jacobi forms on $\mathbf{H}\times C_{\mathbb{C}}$, the product space of the upper half plane and Cayley numbers over the complex field $\mathbb{C}$, was initiated by the author (Eie, 1991) in order to construct the Maaβ space on $\mathcal{H}_2$, the Hermitian upper half plane of degree two over Cayley numbers. They are of particular interest since they are related to modular forms on the exceptional domain developed by Baily (1970, 1973).

In 1993, Kim constructed a singular modular form of weight 4 on the 27 dimensional exceptional domain (Kim, 1993) using the analytic continuation of a non-holomorphic Eisenstein series. Therefore, it is desirable to investigate Jacobi forms on $\mathcal{H}_2\times C_{\mathbb{C}}^2$ more thoroughly since they appear naturally as Fourier-Jacobi coefficients of modular forms on the exceptional domains. Indeed one is able to reconstruct the singular form more easier using the theory of Jacobi forms on $\mathbf{H}\times C_{\mathbb{C}}$ (Krieg, 1997) or on $\mathcal{H}_2\times C_{\mathbb{C}}^2$.

In this paper, we shall first outline the theory of Jacobi forms on $\mathbf{H}\times C_{\mathbb{C}}$ developed by the author and A. Krieg (Eie and Krieg, 1992, 1994). Then, we will proceed to construct a family of Eisenstein series which are examples of Jacobi forms on $\mathcal{H}_2\times C_{\mathbb{C}}^2$. Unlike the case on $\mathbf{H}\times C_{\mathbb{C}}$, we are unable to write down the Eisenstein series explicitly. Instead, we will produce a family of theta series and obtain a group representation ψ_2 from Γ_2 to a unitary group through the transformation formula for the theta series. Here, Γ_2 is a discrete subgroup of the group of bi-holomorphic mappings from $\mathcal{H}_2$ onto itself. In the final sections, we will construct the vector-valued modular forms corresponding to the Eisenstein series using this group representation.

A related problem concerning singular modular forms on $\mathcal{H}_2$ as well as the 27 dimensional exceptional domain will also be investigated. On the Siegel upper half plane, each singular modular form is a linear combination of theta series (Freitag, 1991). However, it is still an open problem to construct theta series on the 27 dimensional exceptional domain since Baily initiated the study of automorphic forms. On $\mathcal{H}_2$, a theta series was constructed in Eie and Krieg (1992) which is a singular modular form of weight 4 as well as a modular form in the Maaβ space.

II. Jacobi Forms over Cayley Numbers

The study of the theory of Jacobi forms began in 1985 with a textbook written by M. Eichler and D. Zagier (Eichler and Zagier, 1985). Let k and m be a pair of non-negative integers. A holomorphic function φ: $\mathbf{H}\times\mathbb{C}\to\mathbb{C}$ is called a Jacobi form of weight k and index m with respect to the modular group $SL_2(\mathbb{Z})$ if it satisfies the following conditions:

(J_1-1) For all $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $SL_2(\mathbb{Z})$,

$$\varphi\left(\frac{az+b}{cz+d}, \frac{w}{cz+d}\right)$$

$$=(cz+d)^k \exp\{2\pi imcw^2/(cz+d)\}\varphi(z,w).$$

(J_1−2) For all integers λ and μ,

$$\varphi(z,w+\lambda z+\mu)=\exp\{-2\pi im(\lambda^2 z+2\lambda w)\}\varphi(z,w).$$

(J_1−3) $\varphi(z,w)$ has a Fourier expansion of the form

$$\varphi(z,w)=\sum_{n=0}^{\infty}\sum_{r\in\mathbb{Z},\ r^2\le 4nm}\alpha(n,r)e^{2\pi i(nz+rw)}.$$

In 1989, Ziegler (Ziegler, 1989) considered Fourier-Jacobi expansions of Siegel modular forms of higher degrees and defined Jacobi forms of several variables on $\mathbf{H}_n\times\mathbb{C}^{nm}$, the product space of the Siegel upper half plane $\mathbf{H}_n$ and the vector space $\mathbb{C}^{nm}$. A. Krieg also gave a general treatment for other kinds of Jacobi forms of several variables (Krieg, 1996).

In 1991, the author introduced the theory of Jacobi forms on $\mathbf{H}\times C_{\mathbb{C}}$, the product space of the upper half plane $\mathbf{H}$ and Cayley numbers over the complex field field, and proved that there is an one-to-one correspondence between modular forms of weight k in the Maaβ space and elliptic modular forms of weight k−4 on the upper half plane. To describe Jacobi forms over Cayley numbers, we need notations concerning Cayley numbers (Eie, 1991).

Let F be a field. The Cayley numbers C_F over F is an eight dimensional vector space over F with a standard basis e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7 satisfying the following rules of multiplication (Baily, 1970):

(1) $xe_0=e_0x=x$ for all x in C_F,

(2) $e_j^2=-e_0$, $j=1, 2, \ldots, 7$,

(3) $e_1e_2e_4=e_2e_3e_5=e_3e_4e_6=e_4e_5e_7=e_5e_6e_1=e_6e_7e_2$
$=e_7e_1e_3=-e_0$.

For $x=\sum_{j=0}^{7}x_je_j$ and $y=\sum_{j=0}^{7}y_je_j$ in C_F, we define the following operations on C_F:

(1) Involution: $x\to\overline{x}=x_0e_0-\sum_{j=1}^{7}x_je_j$,

(2) Trace operator: $T(x)=x+\overline{x}=2x_0$,

(3) Norm operator: $N(x)=x\overline{x}=\overline{x}x=\sum_{j=0}^{7}x_j^2$.

(4) Inner product: σ: $C_F\times C_F\to F$, $\sigma(x,y)=T(x\overline{y})=T(y\overline{x})=2\sum_{j=0}^{7}x_jy_j$.

Note that we have the property: $N(x+y)=N(x)+N(y)+\sigma(x,y)$.

Denote by $\mathbf{o}$ the $\mathbb{Z}$-module in $C_{\mathbb{Q}}$, generated by α_j(j=0, 1, 2, ..., 7), as follows:

$$\alpha_0=e_0,\ \alpha_1=e_1,\ \alpha_2=e_2,\ \alpha_3=-e_4,$$

$$\alpha_4=\frac{1}{2}(e_1+e_2+e_3-e_4),\ \alpha_5=\frac{1}{2}(-e_0-e_1-e_4+e_5),$$

$$\alpha_6=\frac{1}{2}(-e_0+e_1-e_2+e_6),\ \alpha_7=\frac{1}{2}(-e_0+e_2+e_4+e_7).$$

Elements in $\mathbf{o}$ are referred as integral Cayley numbers. This module $\mathbf{o}$ was denoted by J in Coxeter (1946) it satisfies the following conditions:

(1) $N(x)\in\mathbb{Z}$ and $T(x)\in\mathbb{Z}$ for each x in the set,

(2) the set is closed under substraction and multiplication,

(3) the set contains 1.

As shown there, $\mathbf{o}$ is maximal among those modules which have these properties.

Now we are ready to formulate the definition of Jacobi forms on $\mathbf{H}\times C_{\mathbb{C}}$. Let k and m be a pair of non-negative integers. A holomorphic function φ: $\mathbf{H}\times C_{\mathbb{C}}\to\mathbb{C}$ is called a *Jacobi form of weight k and index m* with respect to $SL_2(\mathbb{Z})$ if it satisfies the following conditions:

(J_2−1) For all $\begin{bmatrix} a & b \\ c & d\end{bmatrix}$ in $SL_2(\mathbb{Z})$,

$$\varphi\left(\frac{az+b}{cz+d}, \frac{w}{cz+d}\right)$$

$$=(cz+d)^k \exp\{2\pi imcN(w)/(cz+d)\}\varphi(z,w).$$

(J_2−2) For all λ, μ in $\mathbf{o}$,

$$\varphi(z,w+\lambda z+\mu)=\exp\{-2\pi im[N(\lambda)z+\sigma(\lambda,w)]\}\varphi(z,w).$$

(J_2−3) φ has a Fourier expansion of the form

$$\varphi(z,w)=\sum_{n=0}^{\infty}\sum_{t\in\mathbf{o},\ nm\ge N(t)}\alpha(n,t)e^{2\pi i[nz+\sigma(t,w)]}.$$

III. Examples of Jacobi Forms over Cayley Numbers

Natural examples of Jacobi forms over Cayley numbers come from coefficients of Fourier-Jacobi expansions of modular forms of degree two on $\mathcal{H}_2$, the Hermitian upper half plane of degree two over Cayley numbers. More precisely, we have

$$\mathcal{H}_2=\{Z=\begin{bmatrix} x_1 & x_{12} \\ \overline{x}_{12} & x_2\end{bmatrix}+i\begin{bmatrix} y_1 & y_{12} \\ \overline{y}_{12} & y_2\end{bmatrix}$$

$$|x_1,x_2,y_1,y_2\in\mathbb{R},\ x_{12},y_{12}\in C_{\mathbb{R}},\ y_2>0,\ y_1y_2-N(y_{12})>0\}. \quad \mathbf{(1)}$$

Given

$$Z=\begin{bmatrix} z & w \\ \overline{w} & z^* \end{bmatrix} \in \mathcal{H}_2,$$

Z is invertible and

$$-Z^{-1}=\frac{-1}{zz^*-N(w)}\begin{pmatrix} z^* & -w \\ -\overline{w} & z \end{pmatrix} \in \mathcal{H}_2. \qquad \textbf{(2)}$$

Denote by Γ_2 the discontinuous subgroup of the group of bi-holomorphic mappings from $\mathcal{H}_2$ onto itself, which is generated by transformations as follow:

(1) p_B: $Z \to Z+B$, $B=\begin{bmatrix} n & t \\ \overline{t} & m \end{bmatrix}$, $n,m\in \mathbb{Z}$, $t\in \mathbf{o}$,

(2) t_U: $Z \to {}^t\overline{U}ZU$, $U=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ or $\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$, $t\in \mathbf{o}$,

(3) ι: $Z \to -Z^{-1}$.

Let k be an integer. A holomorphic function f: $\mathcal{H}_2 \to \mathbb{C}$ is called a modular form of weight k with respect to Γ_2 if it satisfies the following conditions:

(M.1) $f({}^t\overline{U}ZU+B)=f(Z)$ for all $B=\begin{bmatrix} n & t \\ \overline{t} & m \end{bmatrix}$, $n,m\in \mathbb{Z}$, $t\in \mathbf{o}$,

$$U=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix},\ t\in \mathbf{o}.$$

(M.2) $f(-Z^{-1})=(\det Z)^k f(Z)$.

Here, we will give an example of a modular form of weight 4 on $\mathcal{H}_2$. We use (T,Z) to denote the inner product of two Hermitian matrices T and Z; $(T,Z)= t_1z+t_1z^*+\sigma(t_{12},w)$ if $T=\begin{bmatrix} t_1 & t_{12} \\ \overline{t}_{12} & t_2 \end{bmatrix}$ and $Z=\begin{bmatrix} z & w \\ \overline{w} & z^* \end{bmatrix}$.

Proposition 1. *The function*

$$f_4(Z)=\sum_{\mathbf{h}\in \mathbf{o}^2} e^{2\pi i(\mathbf{h}{}^t\overline{\mathbf{h}},Z)},\ Z\in \mathcal{H}_2, \qquad \textbf{(3)}$$

is a modular form of weight 4 *with respect to* Γ_2.

Proof. For all $B=\begin{bmatrix} n & t \\ \overline{t} & m \end{bmatrix}$, $n,m\in \mathbb{Z}$, $t\in \mathbf{o}$, it is easy to see that

$$f_4(Z+B)=f_4(Z).$$

Also for $U=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ or $\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$, $t\in \mathbf{o}$, we have

$$(\mathbf{h}{}^t\overline{\mathbf{h}}, {}^t\overline{U}ZU)=((U\mathbf{h})({}^t\overline{\mathbf{h}}{}^t\overline{U}),Z).$$

Note that the lattice $\mathbf{o}^2$ is invariant under the transform $\mathbf{h}\to U\mathbf{h}$. Thus, we have

$$f_4({}^t\overline{U}ZU)=f_4(Z).$$

It remains to prove the transformation formula

$$f(-Z^{-1})=(\det Z)^4 f(Z).$$

It is enough to prove that the formula holds for $Z=iY$ since functions of both sides are holomorphic functions in Z. However, it follows directly that the Fourier transform of the function

$$g(X)=e^{-2\pi(X{}^t\overline{X},Y)},\ X\in C_{\mathbb{R}}^2$$

is given by

$$\hat{g}(W)=(\det Y)^{-4}e^{-2\pi(W{}^t\overline{W},Y^{-1})},\ W\in C_{\mathbb{R}}^2$$

as well as the classical well-known Poisson summation formula

$$\sum_{\mathbf{h}\in \mathbf{o}^2} g(\mathbf{h})=\sum_{\mathbf{R}\in \mathbf{o}^2} \hat{g}(\mathbf{k}). \qquad \square$$

Let

$$\Lambda_2=\{B=\begin{bmatrix} n & t \\ \overline{t} & m \end{bmatrix} \mid n,m\in \mathbb{Z},\ t\in \mathbf{o}\}$$

stand for the lattice of integral Hermitian matrices of size 2×2 over Cayley numbers. Suppose that f is a modular form of weight k on $\mathcal{H}_2$; then, f processes a Fourier expansion of the form

$$f(Z)=\sum_{T\in\Lambda_2,\,T\geq 0} \alpha_f(T)\,e^{2\pi i(T,Z)},\ Z\in \mathcal{H}_2, \qquad \textbf{(4)}$$

due to the Koecher effect. A rearrangement yields the Fourier-Jacobi expansion of f:

$$f(Z)=\sum_{m=0}^{\infty} \varphi_m(z,w)\,e^{2\pi imz^*} \qquad \textbf{(5)}$$

with

$$\varphi_m(z,w)=\sum_{n=0}^{\infty}\sum_{t\in\mathbf{o},\,nm\geq N(t)} \alpha_f\begin{pmatrix} n & t \\ \overline{t} & m \end{pmatrix} e^{2\pi i[nz+\sigma(t,w)]}, \qquad \textbf{(6)}$$

where $\varphi_m(z,w)$ is called the *m-th Fourier-Jacobi coefficient* of f.

Proposition 2. *Suppose that f is a modular form of weight k on $\mathcal{H}_2$ with the Fourier-Jacobi expansion*

$$f(Z)=\sum_{m=0}^{\infty}\varphi_m(z,w)\,e^{2\pi i m z^*}.$$

Then, $\varphi_m(z,w)$ is a Jacobi form of weight k and index m.

Proof. It suffices to prove that $\varphi_m(z,w)$ satisfies $(J_2\text{–}1)$ and $(J_2\text{–}2)$. The mapping

$$L:\begin{bmatrix} z & w \\ \overline{w} & z^* \end{bmatrix}\to\begin{bmatrix} \dfrac{az+b}{cz+d} & \dfrac{w}{cz+d} \\ * & z^*-\dfrac{N(w)c}{cz+d} \end{bmatrix},$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\in SL_2(\mathbb{Z}), \tag{7}$$

is an element of Γ_2. From the transformation formula

$$f(L(Z))=(cz+d)^k f(Z),$$

we get

$$\sum_{m=0}^{\infty}\varphi_m\left(\frac{az+b}{cz+d},\frac{w}{cz+d}\right)\exp\{-2\pi i m N(w)c/(cz+d)\}e^{2\pi i m z^*}$$

$$=(cz+d)^k\sum_{m=0}^{\infty}\varphi_m(z,w)\,e^{2\pi i m z^*}.$$

By comparing the coefficients of $e^{2\pi i m z^*}$ on both sides, we get $(J_2\text{–}1)$. On the other hand, f is invariant under the transform

$$Z\to {}^t\overline{U}\,ZU+B$$

with

$$U=\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \text{ and } B=\begin{bmatrix} 0 & \mu \\ \overline{\mu} & 0 \end{bmatrix},\ \lambda,\mu\in\mathbf{o}.$$

This leads to $(J_2\text{–}2)$. Consequently, $\varphi_m(z,w)$ is a Jacobi form of weight k and index m. □

Let

$$\sum_{T\in\Lambda_2}\alpha(T)\,e^{2\pi i(T,Z)}$$

be the Fourier expansion of $f_4(Z)$; then,

$$\alpha\begin{pmatrix} n & t \\ \overline{t} & m \end{pmatrix}=\#\left\{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\in\mathbf{o}^2 \,\middle|\, n=N(\alpha),\ t=\alpha\overline{\beta},\ m=N(\beta)\right\}.$$

In view of $N(\alpha\overline{\beta})=N(\alpha)N(\beta)$, we have $\alpha(T)=0$ if det $T>0$. Also,

$$\alpha({}^t\overline{U}\,TU)=\alpha(T) \text{ for } U=\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix},\ t\in\mathbf{o}.$$

By Lemma 3.2 in Baily (1970), we are able to reduce T to the form

$$\begin{bmatrix} n & 0 \\ 0 & 0 \end{bmatrix},\ n\geq 0$$

by means of a finite number of the transform $T\to {}^t\overline{U}\,TU$ when det $T=0$. It is easy to see that

$$\alpha\begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix}=240\sum_{d|n}d^3 \text{ for } n\geq 1.$$

For $T\in\Lambda_2$, let $\varepsilon(T)$ be the largest positive integer d such that $\frac{1}{d}T\in\Lambda_2$. Then, we have

$$\alpha(T)=240\sum_{d|\varepsilon(T)}d^3 \text{ if } T\neq 0 \text{ and det } T=0. \tag{8}$$

Therefore, the Fourier coefficients of $f_4(Z)$ satisfy the Maaβ condition:

$$\alpha\begin{pmatrix} n & t \\ \overline{t} & m \end{pmatrix}=\sum_{d|(n,m,t)}d^3\alpha\begin{pmatrix} nm/d^2 & t/d \\ \overline{t}/d & 1 \end{pmatrix}. \tag{9}$$

Here, $d|(n,m,t)$ means that d is a positive common divisor of the integers m and n, and $\frac{t}{d}\in\mathbf{o}$. Hence, $f_4(Z)$ is indeed a modular form of weight 4 in the Maaβ space over Cayley number considered by the author (Eie, 1991). Also, $f_4(Z)$ is a singular modular form since $\alpha(T)=0$ for det $T\neq 0$.

Other examples of Jacobi forms are given explicitly by a family of convergent series. Given $q\in\mathbf{o}$ with $N(q)\equiv 0 \pmod m$, we define the Jacobi-Eisenstein series

$$E_{k,m}(z,w;q)$$

$$=\frac{1}{2}\sum_{(c,d)=1}(cz+d)^{-k}\sum_{\lambda\in\Lambda(q)}\exp\{2\pi i m[N(\lambda)\frac{az+b}{cz+d}$$

$$+\sigma(\lambda,\frac{w}{cz+d})-\frac{cN(w)}{cz+d}]\}. \tag{10}$$

Here,

$$\Lambda(q)=\{t+\frac{q}{m}\,|\,t\in\mathbf{o}\} \text{ and } \begin{bmatrix} a\ b \\ c\ d \end{bmatrix}\in SL_2(\mathbb{Z}).$$

The Jacobi-Eisenstein series $E_{k,m}$ was first considered by the author (Eie, 1995) for the case $q=0$ and then extended to general cases by the author and Krieg (Eie and Krieg, 1994). The series used to define $E_{k,m}$ is absolutely convergent for $k>10$; hence, it defines a Jacobi form of weight k and index m as shown in Eie and Krieg (1994). As we shall see in Section V, we are able to determine the vector-valued modular form corresponding to $E_{k,m}$ and to then conclude that $E_{k,m}$ is indeed a Jacobi form of weight k and index m.

IV. Theta Series and a Group Representation

Given a positive integer m and an integral Cayley number q in $\mathbf{o}$, we define the theta series

$$\vartheta_{m,q}(z,w)=\sum_{t\in\Lambda(q)} e^{2\pi i m[N(t)z+\sigma(t,w)]}$$

$$=\sum_{\lambda\in\mathbf{o}} e^{2\pi i m[N(\lambda+\frac{q}{m})z+\sigma(\lambda+\frac{q}{m},w)]}. \tag{11}$$

Directly from the above definition, it is easy to see that

(1) $\vartheta_{m,q}(z+1,w)=e^{2\pi i N(q)/m}\vartheta_{m,q}(z,w)$,
(2) $\vartheta_{m,q}(z,w+\lambda z+\mu)=e^{-2\pi i m[N(\lambda)z+\sigma(\lambda,w)]}\vartheta_{m,q}(z,w)$,
(3) $\vartheta_{m,q_1}(z,w)=\vartheta_{m,q_2}(z,w)$ if $q_1\equiv q_2 \pmod m$.

We are able to decompose a Jacobi form into a finite linear combination of theta series with coefficients which are elliptic modular forms.

Proposition 3. *Let f be a Jacobi form of weight k and index m with the Fourier expansion*

$$f(z,w)=\sum_{n=0}^{\infty}\sum_{t\in\mathbf{o},\, nm\geq N(t)} \alpha_f(n,t)\, e^{2\pi i[nz+\sigma(t,w)]}.$$

Then, f has the unique expression

$$\sum_{q\,\mathbf{o}/m\mathbf{o}} F_q(z)\cdot\vartheta_{m,q}(z,w),$$

where

$$F_q(z)=\sum_{n\geq N(q)/m}\alpha_f(n,q)\, e^{2\pi i[n-N(q)/m]z}.$$

Proof. Note that for all t, $\lambda\in\mathbf{o}$,

$$\alpha_f(n+\sigma(t,\lambda)+mN(\lambda),t+m\lambda)=\alpha_f(n,t).$$

In other words, $\alpha_f(n,t)$ depends only on $t \pmod m$ and $nm-N(t)$. Set $t=q+m\lambda$ with q ranges over a set of representatives of $\mathbf{o}/m\mathbf{o}$ and λ ranges over all integral Cayley numbers in the second summation of f. It follows that

$$f(z,w)$$

$$=\sum_{n\geq N(q+m\lambda)/m}\ \sum_{q:\mathbf{o}/m\mathbf{o}}\sum_{\lambda\in\mathbf{o}}\alpha_f(n,q+m\lambda)\, e^{2\pi i[nz+\sigma(q+m\lambda,w)]}$$

$$=\sum_{n\geq N(q+m\lambda)/m}\ \sum_{q:\mathbf{o}/m\mathbf{o}}\sum_{\lambda\in\mathbf{o}}\alpha_f(n-\sigma(q,\lambda)-mN(\lambda),q)$$

$$e^{2\pi i[nz+\sigma(q+m\lambda,w)]}.$$

Let $n'=n-\sigma(q,\lambda)-mN(\lambda)$ be a new variable in place of n. Then,

$$n'\geq N(q)/m \text{ if and only if } n\geq N(q+m\lambda)/m$$

and

$$f(z,w)=\sum_{q:\mathbf{o}/m\mathbf{o}}\ \sum_{n'\geq N(q)/m}\alpha_f(n',q)\, e^{2\pi i(n'-N(q)/m)z}$$

$$\cdot\sum_{\lambda\in\mathbf{o}} e^{2\pi i m[N(\lambda+\frac{q}{m})z+\sigma(\lambda+\frac{q}{m},w)]}$$

$$=\sum_{q:\mathbf{o}/m\mathbf{o}} F_q(z)\vartheta_{m,q}(z,w).$$

The vector-valued function

$$F(z)={}^t(F_q(z))_{q\cdot\mathbf{o}/m\mathbf{o}},$$

$$F_q(z)=\sum_{n\geq N(q)/m}\alpha_f(n,q)\, e^{2\pi i(n-N(q)/m)z} \tag{12}$$

is called *the vector-valued modular form* corresponding to the Jacobi form f. Its component $F_q(z)$ is a modular form of weight $k-4$ with respect to the principal congruence subgroup

$$\Gamma[m]=\{\gamma\in SL_2(\mathbb{Z})\,|\,\gamma\equiv E_2 \pmod m\}.$$

Consequently, we can realize the vector space of Jacobi forms of weight k and index m as a subspace $[A(k-4,\Gamma[m])]^{m^8}$, where $A(k-4,\Gamma[m])$ is the vector space of modular forms of weight $k-4$ with respect to $\Gamma[m]$.

In the following, we shall prove the transformation formula of $\vartheta_{m,q}(-\frac{1}{z},\frac{w}{z})$ and $\vartheta_{m,q}(z,w)$.

Proposition 4. *Let $\vartheta_{m,q}(z,w)$ be the theta series defined in* **(11)**. *Then,*

$$\vartheta_{m,q}(-\tfrac{1}{z},\tfrac{w}{z})$$

$$=(\tfrac{z}{m})^4 e^{2\pi i mN(w)/z}\sum_{p:\mathbf{o}/m\mathbf{o}} e^{-2\pi i\sigma(q,p)/m}\vartheta_{m,p}(z,w).$$

Proof. It suffices to prove that the equality holds for $z=iy$ and $w=iv$. We have

$$\vartheta_{m,q}(iy,iv)=\sum_{\lambda\in\mathbf{o}} e^{-2\pi m[N(\lambda+\frac{q}{m})y+\sigma(\lambda+\frac{q}{m},v)]} \tag{13}$$

and

$$\vartheta_{m,q}(iy^{-1},\frac{v}{y})$$
$$=\sum_{t\in\mathbf{o}} e^{-2\pi m[N(t+\frac{q}{m})y^{-1}+\sigma(t+\frac{q}{m},-\frac{iv}{y})]}. \tag{14}$$

Let S be the matrix corresponding to the quadratic form in $g_j(j=0, 1, \ldots, 7)$ of $2N(\sum_{j=0}^{7} g_j\alpha_j)$, i.e., $S=(\sigma(\alpha_i,\alpha_j))_{0\le i,j\le 7}$, and let $\hat{q},\hat{v}$ be the representations of q,v with the basis $\alpha_0, \alpha_1, \ldots, \alpha_7$. Then,

$$\vartheta_{m,q}(iy^{-1},\frac{v}{y})$$
$$=e^{-2\pi mN(v)/y}\sum_{\hat{t}\in\mathbb{Z}^8} e^{-\pi mS[\hat{t}+\hat{q}/m-i\hat{v}]y^{-1}} \tag{15}$$

S is a positive definite symmetric integral matrix and det S=1. By direct calculation, the Fourier transformation of the function

$$f(x)=e^{-\pi mS[x+\hat{q}/m-i\hat{v}]y^{-1}}$$

is given by

$$\hat{f}(z)=(\frac{y}{m})^4 e^{-2\pi i\langle\hat{q}/m-i\hat{v},z\rangle} e^{-\pi yS^{-1}[z]/m}.$$

Here, $<\alpha,\beta>$ is the inner product of α, β in $\mathbb{R}^8$. Then, the Poisson summation formula implies that

$$\vartheta_{m,q}(iy^{-1},\frac{v}{y})$$
$$=e^{-2\pi mN(v)/y}(\frac{y}{m})^4\sum_{z\in\mathbb{Z}^8} e^{-2\pi i\langle\hat{q}/m-i\hat{v},z\rangle} e^{-\pi yS^{-1}[z]/m}$$
$$=e^{-2\pi mN(v)/y}(\frac{y}{m})^4\sum_{t\in\mathbf{o}} e^{-2\pi i\sigma(q/m-iv,t)} e^{-2\pi yN(t)/m}. \tag{16}$$

Now, letting $t=p+m\lambda$ with p ranges over all coset representatives of $\mathbf{o}/m\mathbf{o}$ and λ ranges over all integral numbers, we get our assertion. □

Fix a set of representatives $q_1, q_2, \ldots, q_{m^8}$ of $\mathbf{o}/m\mathbf{o}$ and let

$$\Theta={}^t(\vartheta_{m,q_1},\vartheta_{m,q_2},\ldots,\vartheta_{m,q_{m^8}}).$$

Proposition 5. *There exists a unique group homomorphism ψ: $SL_2(\mathbb{Z})\to U(m^8)$ such that*

$$\Theta(\frac{az+b}{cz+d},\frac{w}{cz+d})$$
$$=(cz+d)^4 e^{2\pi icN(w)/(cz+d)}\overline{\psi\left(\begin{bmatrix} a & b\\ c & d\end{bmatrix}\right)}\Theta(z,w)$$

for all $\begin{bmatrix} a & b\\ c & d\end{bmatrix}\in SL_2(\mathbb{Z})$. *In particular, we have*

(a) $\psi(T)=diag(e^{-2\pi iN(q_1)/m},\ \ldots,\ e^{-2\pi iN(q_{m^8})/m})$, $T=\begin{bmatrix} 1 & 1\\ 0 & 1\end{bmatrix}$,

(b) $\psi(J)=m^{-4}(e^{2\pi i\sigma(q_\nu,q_\mu)/m})_{\nu,\mu=1,\ldots,m^8}$, $J=\begin{bmatrix} 0 & -1\\ 1 & 0\end{bmatrix}$.

Proof. $\psi(T)$ and $\psi(J)$ are determined from the definition and the previous proposition. Note that $SL_2(\mathbb{Z})$ is generated by T and J. This proves our assertion. □

Now, we shall employ an argument similar to that in Chapter IX of Schoenberg (1974) to obtain the explicit expression of ψ from $\psi(T)$ an $\psi(J)$.

Lemma 1. *For any interger $c\neq 0$, we have*

$$\vartheta_{m,q}(z,w)=\sum_{\lambda:\mathbf{o}/c\mathbf{o}}\vartheta_{mc,q+m\lambda}(cz,w).$$

Proof. From the definition

$$\vartheta_{m,q}(z,w)=\sum_{t\in\mathbf{o}}\exp\{2\pi im[N(t+\frac{q}{m})z+\sigma(t+\frac{q}{m},w)\},$$

we let $t=\lambda+cp$ with λ ranges over all coset representatives of $\mathbf{o}/c\mathbf{o}$ and p ranges over all integral Cayley numbers, and get the identity. □

Proposition 6. *For* $\begin{bmatrix} a & b\\ c & d\end{bmatrix}\in SL_2(\mathbb{Z})$ *with $c\neq 0$, we have*

$$\psi_{q,p}\left(\begin{bmatrix} a & b\\ c & d\end{bmatrix}\right)=(mc)^{-4}\sum_{\lambda:\mathbf{o}/c\mathbf{o}}\exp\{\frac{-2\pi i}{mc}[aN(q+m\lambda)$$
$$-\sigma(q+m\lambda,p)+dN(p)]\}.$$

Proof. By Lemma 1, we get

$$\vartheta_{m,q}(\frac{az+b}{cz+d},\frac{w}{cz+d})$$
$$=\sum_{\lambda:\mathbf{o}/c\mathbf{o}}\vartheta_{mc,q+m\lambda}(\frac{c(az+b)}{cz+d},\frac{w}{cz+d})$$
$$=\sum_{\lambda:\mathbf{o}/c\mathbf{o}}\vartheta_{mc,q+m\lambda}(a-\frac{1}{cz+d},\frac{w}{cz+d})$$
$$=\sum_{\lambda:\mathbf{o}/c\mathbf{o}}\exp\{2\pi iaN(q+m\lambda)/mc\}$$

$$\cdot\, \vartheta_{mc,q+m\lambda}(-\frac{1}{cz+d}, \frac{w}{cz+d}).$$

In the other hand,

$$\vartheta_{mc,q+m\lambda}(-\frac{1}{cz+d}, \frac{w}{cz+d})$$

$$=(cz+d)^4(mc)^{-4}\exp\{2\pi iN(w)c/(cz+d)\}$$

$$\sum_{p:o/mco}\exp\{2\pi i\,\sigma(q+m\lambda,p)/mc\}\vartheta_{mc,p}(cz+d, w).$$

Also, we have

$$\vartheta_{mc,p}(cz+d,w)=\exp\{2\pi idN(p)/mc\}\vartheta_{mc,p}(cz,w).$$

Hence, we get

$$\vartheta_{m,q}(-\frac{az+b}{cz+d}, \frac{w}{cz+d})$$

$$=(cz+d)^4(mc)^{-4}\exp\{2\pi iN(w)c/(cz+d)\}$$

$$\times\sum_{p:o/mco}\vartheta_{mc,p}(cz,w)\sum_{\lambda:o/co}\exp\{\frac{2\pi i}{mc}[aN(q+m\lambda)$$

$$-\sigma(q+m\lambda,p)+dN(p)]\}.$$

Denote by $S(p,q)$ the sum in the second summation. If we can prove that $S(p+m\mu,q)=S(p,q)$, then our assertion will follow since

$$\sum_{p:o/mco}\vartheta_{mc,p}(cz,w)S(p,q)$$

$$=\sum_{p:o/mco}\sum_{\mu:o/co}\vartheta_{mc,p+m\mu}(cz,w)\,S(p+m\mu,q)$$

$$=\sum_{p:o/co}\vartheta_{m,p}(z,w)\,S(p,q).$$

Rewrite $S(p,q)$ as

$$\exp\{\frac{2\pi i}{mc}[aN(q)-\sigma(q,p)]\}$$

$$\sum_{\lambda:o/co}\exp\{\frac{2\pi i}{mc}[am^2N(\lambda)-m\sigma(\lambda,p-aq)+dN(p)]\}.$$

Since $(c,d)=1$, we can replace λ with $d\lambda$ in the summation. Therefore,

$$S(p,q)=\exp\{\frac{2\pi i}{mc}[aN(q)-\sigma(q,p)]\}$$

$$\sum_{\lambda:o/co}\exp\{\frac{2\pi id}{mc}[m^2N(\lambda)-m\sigma(\lambda,p-aq)+N(p)]\}$$

$$=\exp\{\frac{2\pi i}{mc}[b\,\sigma(p,q)-abN(q)]\}$$

$$\sum_{\lambda:o/mo}\exp\{\frac{2\pi imd}{c}N(\lambda-\frac{p-aq}{m})\}.$$

From the above, we can see that $S(p,q)$ is invariant under the transform $p\to p+m\mu$; hence, our proof is complete. □

V. Jacobi Forms as Vector-Valued Modular Forms

As shown in Proposition 3, we are able to decompose a Jacobi form into an inner product of a vector-valued modular form and the vector of theta series in **(11)**. Here, we will discuss the necessary and sufficient conditions for a vector-valued modular form corresponding to a Jacobi form.

Proposition 7. *Let $q_1, q_2, \ldots, q_{m^8}$ be a set of representatives of* o/mo *and*

$$F(z)={}^t(F_{q_1}(z), F_{q_2}(z), \ldots, F_{q_{m^8}}(z))$$

with

$$F_q(z)=\sum_{n\geq N(q)/m}\alpha(n,q)\,e^{2\pi i(n-N(q)/m)z}.$$

Then, the following statements are equivalent:

(A) $f(z,w)={}^tF(z)\cdot\Theta(z,w)$ *is a Jacobi form of weight k and index m.*

(B) $F(\frac{az+b}{cz+d})=(cz+d)^{k-4}\psi\left(\begin{bmatrix} a & b \\ c & d\end{bmatrix}\right)F(z)$ *for all* $\begin{bmatrix} a & b \\ c & d\end{bmatrix}\in SL_2(\mathbb{Z})$.

Proof. Given a Jacobi form f of weight k and index m and $\begin{bmatrix} a & b \\ c & d\end{bmatrix}\in SL_2(\mathbb{Z})$, we have

$${}^tF(\frac{az+b}{cz+d})\cdot\Theta(\frac{az+b}{cz+d}, \frac{w}{cz+d})$$

$$=(cz+d)^k e^{2\pi imcN(w)/(cz+d)}\,{}^tF(z)\cdot\Theta(z,w).$$

By Proposition 5, we get

$${}^tF(\frac{az+b}{cz+d})\cdot\overline{\psi\left(\begin{bmatrix} a & b \\ c & d\end{bmatrix}\right)}\Theta(z,w)$$

$$=(cz+d)^{k-4}\,{}^tF(z)\cdot\Theta(z,w).$$

Since the components of the vector-valued theta function $\Theta(z,w)$ are linearly independent and $\psi\left(\begin{bmatrix} a & b \\ c & d\end{bmatrix}\right)$

is unitary, we conclude that

$$F(\frac{az+b}{cz+d})=(cz+d)^{k-4}\psi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)F(z).$$

This proves that (A) implies (B). The converse is a direct verification. □

Let $\Gamma_1=SL_2(\mathbb{Z})$ and

$$\Gamma_1^\infty=\left\{\pm\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}\middle| n\in\mathbb{Z}\right\}. \quad \textbf{(17)}$$

With the transformation formula in Proposition 4, we can rewrite the Jacobi-Eisenstein series $E_{k,m}(z,w;q)$ in **(10)** as

$$E_{k,m}(z,w;q)=\sum_{M:\Gamma_1/\Gamma_1^\infty} j(M,z)^{4-k}\sum_{p:o/mo}\overline{\psi_{q,p}(M)}\vartheta_{m,p}(z,w). \quad \textbf{(18)}$$

Here, $j(M,z)=(cz+d)$ if $M=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and

$$\psi(M)=[\psi_{p,q}(M)]_{p,q:o/mo}.$$

Suppose that

$$E_{k,m}(z;q)={}^t(E_{q_1}(z;q),\ \ldots,\ E_{q_{m^8}}(z;q))$$

is the vector-valued modular form corresponding to $E_{k,m}(z,w;q)$. Then,

$$E_p(z;q)=\sum_{M:\Gamma_1/\Gamma_1^\infty}\overline{\psi_{q,p}(M)}j(M,z)^{4-k}. \quad \textbf{(19)}$$

It is clear that the above series is absolutely convergent for $k\geq 7$. In the following, we will prove that $E_{k,m}(z;q)$ is indeed a vector-valued modular form corresponding to a Jacobi form.

Proposition 8. *For all $K\in\Gamma_1=SL_2(\mathbb{Z})$, we have*

$$E_{k,m}(K(z);q)=j(K,z)^{k-4}\psi(K)E_{k,m}(z;q)$$

if $N(q)\equiv 0$ (mod m).

Proof. Consider the matrix of modular functions defined by

$$G(z)=\sum_{M:\Gamma_1/\Gamma_1^\infty}\overline{\psi(M)}j(M,z)^{4-k},\ k\geq 7.$$

The function G depends on the choice of the coset representatives of Γ_1/Γ_1^∞. Indeed, we have

$$\psi\left(\begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}\right)=\psi(T)\psi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right).$$

However, its q-th row is independent of the choice since

$$\psi(T)=\mathrm{diag}[e^{-2\pi iN(q_1)/m},\ \ldots,\ e^{-2\pi iN(q_{m^8})/m}]$$

and $N(q)\equiv 0$ (mod m). Note that ${}^tE_{k,m}(z;q)$ is precisely the q-th row of $G(z)$. From the group properties of Γ_1 and the cocycle condition of j,

$$j(M,K(z))=j(MK,z)j(K,z)^{-1},$$

we conclude that

$${}^tE_{k,m}(K(z);q)=j(K,z)^{k-4}\,{}^tE_{k,m}(z,q)\overline{\psi(K^{-1})}$$

for all $K\in\Gamma_1$. Since $\psi(K)$ is unitary, it follows that

$$E_{k,m}(K(z);q)=j(K,z)^{k-4}\psi(K)E_{k,m}(z;q).$$ □

Proposition 9. *For $k\geq 7$, the Fourier expansion of $E_p(z;q)$ is given by*

$$E_p(z;q)=[\psi_{q,p}(E)+(-1)^k\psi_{q,p}(-E)] + \sum_{n>N(p)/m} a(n,p)e^{2\pi i(n-N(p)/m)z}$$

with

$$a(n,p)=\frac{(-2\pi i)^{k-4}}{(k-5)!\,m}(n-N(p)/m)^{k-5}$$

$$\sum_{c\neq 0}c^{4-k}\sum_{\substack{1\leq d<m|c|\\(d,c)=1}}\overline{\psi_{q,p}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)}$$

$$\exp\{2\pi i(mn-N(p)d)/mc\}.$$

Proof. According to $c=0$ or not, we have

$$E_p(z;q)=[\psi_{q,p}(E)+(-1)^k\psi_{q,p}(-E)] + \sum_{c\neq 0}c^{4-k}\sum_{(c,d)=1}\overline{\psi_{q,p}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)}(z+d/c)^{4-k}.$$

Let $d=d'+lmc$ with $1\leq d'<m|c|$, $(d',c)=1$ and $l\in\mathbb{Z}$. Also, note that

$$\psi_{q,p}\left(\begin{bmatrix} a & b+lma \\ c & d+lmc \end{bmatrix}\right)=\psi_{q,p}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right).$$

It follows that

$$\sum_{(c,d)=1} \overline{\psi_{q,p}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)}(z+d/c)^{4-k}$$

$$= m^{4-k} \sum_{\substack{1 \le d' < m|c| \\ (d;c)=1}} \overline{\psi_{q,p}\left(\begin{bmatrix} a & b' \\ c & d' \end{bmatrix}\right)} \sum_{l \in \mathbb{Z}} \left(\frac{z}{m} + \frac{d'}{mc} + l\right)^{4-k}$$

$$= \frac{m^{4-k}(-2\pi i)^{k-4}}{(k-5)!} \sum_{n=1}^{\infty} n^{k-5} e^{2\pi i n z/m}$$

$$\sum_{\substack{1 \le d' < m|c| \\ (d',c)=1}} \overline{\psi_{q,p}\left(\begin{bmatrix} a & b' \\ c & d' \end{bmatrix}\right)} e^{2\pi i n d'/mc}.$$

On the other hand, by Proposition 8, we have

$$E_p(z+1;q) = e^{-2\pi i N(p)/m} E(z;q).$$

This forces the coefficient of $e^{2\pi i z/m}$ to be zero unless $n+N(p)\equiv 0 \pmod m$. Let $n'=[n+N(p)]/m$ be a new variable in place of n. Then,

$$\sum_{(c,d)=1} \overline{\psi_{q,p}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)}(z+d/c)^{4-k}$$

$$= \frac{(-2\pi i)^{k-4}}{(k-5)!\,m} \sum_{n' > N(p)/m} (n' - N(p)/m)^{k-5} e^{2\pi i (n' - N(p)/m) z}$$

$$\times \sum_{\substack{1 \le d' < m|c| \\ (d',c)=1}} \overline{\psi_{q,p}\left(\begin{bmatrix} a & b' \\ c & d' \end{bmatrix}\right)} \exp[2\pi i (mn' - N(p))\, d'/mc].$$

This proves our assertion on the Fourier coefficient $a(n,p)$ of $E_p(z;q)$. □

From the definition of $E_{k,m}(z,w;q)$, we can see that
(1) $E_{k,m}(z,w;q)=E_{k,m}(z,w;q')$ if $q\equiv q' \pmod m$,
(2) $E_{k,m}(z,w;-q)=E_{k,m}(z,-w;q)=(-1)^k E_{k,m}(z,w;q)$.
Thus $E_{k,m}(z,w;q)=0$ if and only if k is odd and $2q\equiv 0 \pmod m$. Fix a set of representatives $\pm q_1, \pm q_2, \dots, \pm q_r, q_{r+1}, \dots, q_{r+s}$ of the set

$$\{q \in \mathbf{o}/m\mathbf{o} \mid N(q)\equiv 0 \pmod m\}$$

such that $2q_j \notin m\mathbf{o}$ for $1\le j\le r$ and $2q_j \in m\mathbf{o}$ for $r+1\le j\le r+s$. Then, the set

$$S = \begin{cases} \{E_{k,m}(z;w;\pm q_j) \mid 1 \le j \le r\}, \text{ if } k \text{ is odd}, \\ \{E_{k,m}(z;w;\pm q_j) \mid 1 \le j \le r+s\}, \text{ if } k \text{ is even}; \end{cases}$$

are linearly independent sets. Thus, the number of independent Jacobi-Eisenstein series is

$$\frac{1}{2}\left(m^7 \sum_{d|m} \frac{\varphi(d)}{d^4} + (-1)^k N_m\right),$$

where

$$N_m = \begin{cases} 1 \text{ if } m \equiv 1 \pmod 2) \\ 136 \text{ if } m \equiv 2 \pmod 4) \\ 256 \text{ if } m \equiv 0 \pmod 4). \end{cases}$$

Remark. The formula

$$|\{q \in \mathbf{o}/m\mathbf{o} \mid N(q)\equiv 0 \text{ (mode } m)\}| = m^7 \sum_{d|m} \frac{\varphi(d)}{d^4}$$

follows as a corollary of Karel (1974).

VI. Modular Forms on the Exceptional Domain

Let $\mathcal{J}_{\mathbb{R}}$ be the set of 3×3 Hermition matrices over real Cayley numbers. $\mathcal{J}_{\mathbb{R}}$ consists of matrices of the following form:

$$X = \begin{bmatrix} \xi_1 & x_{12} & x_{13} \\ \overline{x}_{12} & \xi_2 & x_{23} \\ \overline{x}_{13} & \overline{x}_{23} & \xi_3 \end{bmatrix}, \ \xi_1, \xi_2, \xi_3 \in \mathbb{R}, x_{12}, x_{13}, x_{23} \in C_{\mathbb{R}}, \quad \textbf{(20)}$$

For $X \in \mathcal{J}_{\mathbb{R}}$ as given in **(20)**, we define
(1) $tr(X)=\xi_1+\xi_2+\xi_3$,
(2) $\det(X)=\xi_1\xi_2\xi_3-\xi_1 N(x_{23})-\xi_2 N(x_{13})-\xi_3 N(x_{12})+T((x_{12}x_{23})\overline{x}_{13})$,
(3) $X\times X = X^2 - tr(X)X + \frac{1}{2}(tr(X)^2 - tr(X^2))E$

$$= \begin{bmatrix} \xi_2\xi_3 - N(x_{23}) & x_{13}\overline{x}_{23} - \xi_3 x_{12} & x_{12}x_{23} - \xi_2 x_{13} \\ x_{23}\overline{x}_{13} - \xi_3 \overline{x}_{12} & \xi_1\xi_3 - N(x_{13}) & \overline{x}_{12}x_{13} - \xi_1 x_{23} \\ \overline{x}_{23}\,\overline{x}_{12} - \xi_2 \overline{x}_{13} & \overline{x}_{13}x_{12} - \xi_1 \overline{x}_{23} & \xi_1\xi_2 - N(x_{12}) \end{bmatrix}.$$

Note that X is invertible if and only if $\det X \neq 0$. In this case, the inverse is given by

$$X^{-1} = \frac{1}{\det X}(X \times X).$$

Also, we set

rank $X=1$ if and only if $X\neq 0$, $X\times X=0$,

rank $X=2$ if and only if $X\times X\neq 0$, $\det X=0$,

rank $X=3$ if and only if $\det X\neq 0$.

We supply $\mathcal{J}_{\mathbb{R}}$ with a product defined by

$$X \circ Y = \frac{1}{2}(XY+YX), \tag{21}$$

where XY is the ordinary matrix product. Then, $\mathcal{J}_{\mathbb{R}}$ becomes a real Jordan algebra with this product. Define an inner product on $\mathcal{J}_{\mathbb{R}}$ by

$$(X,Y)=tr(X \circ Y). \tag{22}$$

Finally, we let $\mathfrak{R}$ be the set of squares $X \circ X$ of the elements of $\mathcal{J}_{\mathbb{R}}$ and $\mathfrak{R}^+$ be the interior of $\mathfrak{R}$. The exceptional domain in $\mathbb{C}^{27}$ is, then, defined by

$$\mathcal{H}=\{Z=X+iY|X,Y\in \mathcal{J}_{\mathbb{R}}, Y\in \mathfrak{R}^+\}. \tag{23}$$

Set $\mathcal{J}_o=\mathcal{J}_{\mathbb{R}} \cap M_3(\mathbf{o})$. Here, $M_3(\mathbf{o})$ is the set of 3×3 martices over the integral Cayley numbers. For $1\le i$, $j\le 3$, let e_{ij} be the 3×3 matrix with 1 at the ij-position and 0 elsewhere. When $i\neq j$, $t\in C_{\mathbb{R}}$, we let $U_{ij}(t)=E+te_{ij}$, where E is the 3×3 identity matrix.

The group of holomorphic automorphisms $\mathcal{G}$ of $\mathcal{H}$ is a Lie group of type E_7 (Baily, 1970). Let Γ be the discrete subgroup of $\mathcal{G}_{\mathbb{R}}$ generated by the following automorphisms of $\mathcal{H}$:

(1) ι: $Z\to -Z^{-1}$,

(2) p_B: $Z\to Z+B$, $B\in \mathcal{J}_o$,

(3) t_U: $Z\to Z[U]={}^tUZU$, $U=U_{ij}(t)$, $t\in \mathbf{o}$.

Let k be an even integer. A holomorphic function f defined on $\mathcal{H}$ is a modular form of weight k with respect to Γ if it satisfies the following conditions:

(a) $f(-Z^{-1})=(\det(-Z))^k f(Z)$,

(b) $f(Z[U]+B)=f(Z)$ for all $B\in \mathcal{J}_o$ and $U=U_{ij}(t)$, $t\in \mathbf{o}$.

In particular, from (b), a modular from f on $\mathcal{H}$ has a Fourier expansion of the form

$$f(Z)=\sum_{T\in \mathfrak{R}\cap \mathcal{J}_0} a(T)\, e^{2\pi i(T,Z)}.$$

f is a *singular modular form* if $a(T)=0$ unless $\det T=0$.

Baily (1970) considered the Eisenstein series

$$E_l(Z)=\sum_{r\in \Gamma/\Gamma_0} \mathbf{j}(\gamma,Z)^l,\ Z\in \mathcal{H}. \tag{24}$$

Here Γ_0 is the subgroup of Γ generated by p_B, t_U with $B\in \mathcal{J}_o$, $U=U_{ij}(t)$, $t\in \mathbf{o}$. $\mathbf{j}(\gamma,Z)$ is the determinant of the Jacobian matrix of γ at Z, and it has the following properties:

(1) $\mathbf{j}(p_B,Z)=1$ for all $B\in \mathcal{J}_{\mathbb{R}}$,

(2) $\mathbf{j}(t_U,Z)=1$ for all $U=U_{ij}(t)$, $t\in \mathbf{o}$,

(3) $\mathbf{j}(\iota,Z)=[\det(-Z)]^{-18}$.

For any positive even integer, the series in **(24)** converges absolutely and uniformly on any compact subset of $\mathcal{H}$. Hence, E_l is a modular form of weight $18l$ with respect to Γ on $\mathcal{H}$ and has a Fourier expansion:

$$E_l(Z)=\sum_{T\in \mathcal{J}_0\cap \mathfrak{R}} a_l(T)\, e^{2\pi i(T,Z)}. \tag{25}$$

Baily proved that the Fourier coefficients $a_l(T)$ of $E_l(Z)$ are rational numbers and concluded that the Satake compactification of $\mathcal{H}/\Gamma$ has a biregularly equivalent projective model defined over the rational number field $\mathbf{Q}$ (Baily, 1970).

VII. Jacobi Forms of Degree Two over Cayley Numbers

By a *Jacobi form of degree two over Cayley numbers*, we mean a Jacobi form defined on $\mathcal{H}_2\times C^2_{\mathbb{C}}$. Let k and $m\ge 1$ be non-negative integers. A holomorphic function f: $\mathcal{H}_2\times C^2_{\mathbb{C}}\to \mathbb{C}$ is called *a Jacobi form of weight k and index m* with respect to Γ_2 if f satisfies the following conditions:

(J-1) $f(Z+B,W)=f(Z,W)$ for all $B\in \Lambda_2$,

(J-2) $f(Z[U],{}^t\overline{U}\,W)=f(Z,W)$ for all $U=\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $t\in \mathbf{o}$,

(J-3) $f(-Z^{-1}, Z^{-1}W)=(\det Z)^k \exp\{2\pi i m Z^{-1}[W]\} f(Z,W)$,

(J-4) $f(Z,W+Z\mathbf{q}+\mathbf{p})=\exp\{-2\pi i m[(Z,\mathbf{q}\,{}^t\overline{\mathbf{q}})+\sigma(w_1, q_1)+\sigma(w_2, q_2)]\} f(Z,W)$=for all $\mathbf{q}={}^t(q_1, q_2)$, $\mathbf{p}=(p_1, p_2)\in \mathbf{o}^2$,

(J-5) f has a Fourier expansion of the form,

$$f(Z,W) = \sum_{\mathbf{q}\in \mathbf{o}^2}\ \sum_{T\in \Lambda_2,\, T\ge \mathbf{q}\,{}^t\overline{\mathbf{q}}/m} a(T,\mathbf{q})\, e^{2\pi i[(T,Z)+\sigma(q_1,w_1)+\sigma(q_2,w_2)]}. \tag{26}$$

Here, for $Z=\begin{bmatrix} z_1 & z_{12} \\ \overline{z}_{12} & z_2 \end{bmatrix}\in \mathcal{H}_2$ and $W=\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\in C^2_{\mathbb{C}}$, we let

$$Z[W]=z_1N(w_1)+z_2N(w_2)+\sigma(z_{12},w_1\overline{w}_2).$$

For a 2×2 Hermitian matrix $A=\begin{bmatrix} a & \lambda \\ \overline{\lambda} & b \end{bmatrix}$, we write $A\ge 0$ to mean $a\ge 0$ and $ab\ge N(\lambda)$. Also, $A\ge B$ if and only if $A-B\ge 0$.

From the above definition, we are able to decompose a Jacobi form of degree two into an inner product of a vector-valued modular form and a vector-valued theta series.

Proposition 10. *Let $f(Z,W)$ be a Jacobi form of degree two with Fourier expansion (26). Then,*

$$f(Z,W)=\sum_{\mathbf{q}:(\mathbf{o}/m\mathbf{o})^2} F_{\mathbf{q}}(Z)\vartheta_{m,\mathbf{q}}(Z,W),$$

where

$$F_{\mathbf{q}}(Z)=\sum_{T\geq \mathbf{q}^t\overline{\mathbf{q}}/m} a(T,\mathbf{q})\,e^{2\pi i(T-\mathbf{q}^t\overline{\mathbf{q}}/m,Z)}$$

and

$$\vartheta_{m,\mathbf{q}}(Z,W)$$

$$=\sum_{\mathbf{h}=\lambda+\mathbf{q}/m,\ \lambda\in\mathbf{o}^2}\exp\{2\pi im[(\mathbf{h}^t\overline{\mathbf{h}},Z)+\sigma(h_1,w_1)+\sigma(h_2,w_2)]\}.$$

Proof. Set $\mathbf{p}=\mathbf{q}+m\lambda$ with $\mathbf{q}$ ranges over all representatives of $(\mathbf{o}/m\mathbf{o})^2$ and λ ranges over $\mathbf{o}^2$ in the first summation of $f(Z,W)$. Then, our assertion follows from (J-4) and direct verification. □

For each $\mathbf{q}={}^t(q_1,q_2)\in\mathbf{o}^2$, consider the theta series $\vartheta_{m,\mathbf{q}}(Z,W)$ defined by

$$\vartheta_{m,\mathbf{q}}(Z,W)$$

$$=\sum_{\mathbf{h}=\lambda+\mathbf{q}/m,\ \lambda\in\mathbf{o}^2}\exp\{2\pi im[(\mathbf{h}^t\overline{\mathbf{h}},Z)+\sigma(h_1,w_1)+\sigma(h_2,w_2)]\}. \quad \textbf{(27)}$$

Obviously, we have

$$\vartheta_{m,\mathbf{q}}(Z+B,W)=e^{2\pi im(\mathbf{q}^t\overline{\mathbf{q}},B)/m}\vartheta_{m,\mathbf{q}}(Z,W)$$

for all $B\in\Lambda_2$, **(28)**

$$\vartheta_{m,\mathbf{q}}(Z[U],{}^t\overline{U}\,W)=\vartheta_{m,U\mathbf{q}}(Z,W),$$

for $U=\begin{bmatrix}1&0\\t&1\end{bmatrix}$ or $\begin{bmatrix}0&1\\-1&0\end{bmatrix}$, $t\in\mathbf{o}$. **(29)**

Here, we shall prove the transformation formula between $\vartheta_{m,\mathbf{q}}(-Z^{-1}, Z^{-1}W)$ and $\vartheta_{m,\mathbf{q}}(Z,W)$. We need the following.

Lemma 2. *For each* $\mathbf{h}={}^t(h_1,h_2)\in C_{\mathbb{R}}^2$, $\Lambda=diag[\xi_1,\xi_2]$, $\xi_1>0$, $\xi_2>0$, *we have*

$$(\mathbf{h}^t\overline{\mathbf{h}},\Lambda[U])=((U\mathbf{h})({}^t\overline{\mathbf{h}}\,{}^t\overline{U}),\Lambda)$$

for all $U=\begin{bmatrix}1&t\\0&1\end{bmatrix}$, $\begin{bmatrix}1&0\\t&1\end{bmatrix}$ *or* $\begin{bmatrix}0&1\\-1&0\end{bmatrix}$, $t\in C_{\mathbb{R}}$.

Proof. It is obvious for $U=\begin{bmatrix}0&1\\-1&0\end{bmatrix}$. Here, we will prove the case $U=\begin{bmatrix}1&t\\0&1\end{bmatrix}$. We have

$$(\mathbf{h}^t\overline{\mathbf{h}},\Lambda[U])=\xi_1N(h_1)+\xi_2\sigma(t,h_1\overline{h_2})$$

$$+(\xi_2+\xi_1N(t))N(h_2).$$

On the other hand,

$$U\mathbf{h}=\begin{bmatrix}h_1+th_2\\h_2\end{bmatrix}.$$

It follows that

$$((U\mathbf{h})({}^t\overline{\mathbf{h}}\,{}^t\overline{U}),\ \Lambda)=\xi_1N(h_1+th_2)+\xi_2N(h_2)$$

$$=\xi_1N(h_1)+\xi_1\sigma(h_1,th_2)$$

$$+(\xi_1N(t_1)+\xi_2)N(h_2).$$

Hence, our assertion follows form the fact that

$$\sigma(t,h_1\overline{h_2})=T(t(h_2\overline{h_1}))$$

$$=T((th_2)\overline{h_1})$$

$$=\sigma(h_1,th_2). \quad □$$

In exactly the same way, we can prove the following.

Lemma 3. *For* $\mathbf{h}={}^t(h_1,h_2)\in C_{\mathbb{R}}^2$, $\mathbf{V}={}^t(v_1,v_2)\in C_{\mathbb{R}}^2$, $\Lambda=diag[\xi_1,\xi_2]$, $\xi_1>0$, $\xi_2>0$ *and* $U=\begin{bmatrix}1&t\\0&1\end{bmatrix}$ or $\begin{bmatrix}1&0\\t&1\end{bmatrix}$, $t\in C_{\mathbb{R}}$, *we have*

$$T({}^t\overline{\mathbf{h}}(\Lambda^{-1}[{}^t\overline{U}^{-1}])V)=T({}^t\overline{\mathbf{h}}U^{-1})(\Lambda^{-1}({}^t\overline{U}^{-1}\mathbf{V})).$$

Proposition 11. *Suppose that* $\vartheta_{m,\mathbf{q}}(Z,W)$ *is defined as in* **(27)**. *Then,*

$$\vartheta_{m,\mathbf{q}}(-Z^{-1}, Z^{-1}W)$$

$$=(\det Z)^4\exp\{2\pi imZ^{-1}[W]\}$$

$$\times\frac{1}{m^8}\sum_{\mathbf{p}:(\mathbf{o}/m\mathbf{o})^2}e^{-2\pi i[\sigma(q_1,p_1)+\sigma(q_2,p_2)]/m}\vartheta_{m,\mathbf{p}}(Z,W). \quad \textbf{(30)}$$

Proof. It suffices to prove that **(30)** holds for $Z=iY$ and $W=iV$ since both sides are holomorphic functions in Z and W. Let $Y=\Lambda[U]$ with $\Lambda=\mathrm{diag}[\xi_1, \xi_2]$ and $U=\begin{bmatrix}1&t\\0&1\end{bmatrix}$. Then,

$$Y^{-1}=\Lambda^{-1}[{}^tU^{-1}].$$

It follows that

$$\vartheta_{m,\mathbf{q}}(iY^{-1}, Y^{-1}V)$$

$$= \sum_{\mathbf{h}=\lambda+\mathbf{q}/m,\ \lambda\in\mathbf{o}^2} \exp\{-2\pi m(\mathbf{h}^t\overline{\mathbf{h}}, Y^{-1})$$

$$+2\pi i m T({}^t\overline{\mathbf{h}}(Y^{-1}V))\}$$

$$= \sum_{\mathbf{h}=\lambda+\mathbf{q}/m,\ \lambda\in\mathbf{o}^2} \exp\{-2\pi m(\mathbf{h}^t\overline{\mathbf{h}}, \Lambda^{-1}[{}^tU^{-1}]$$

$$+2\pi i m T(({}^t\overline{h})(\Lambda^{-1}[{}^t\overline{U}^{-1}]V))\}$$

$$= \sum_{\mathbf{h}=\lambda+\mathbf{q}/m,\ \lambda\in\mathbf{o}^2} \exp\{-2\pi m(({}^tU^{-1}\mathbf{h})({}^t\overline{\mathbf{h}}U^{-1}), \Lambda^{-1})$$

$$+2\pi i m T(({}^t\overline{\mathbf{h}}U^{-1})\Lambda^{-1}({}^t\overline{U}^{-1}V))\}$$

(By Lemma 2 and 3)

$$= e^{-2\pi m\Lambda^{-1}[{}^t\overline{U}^{-1}V]}(\xi_1\xi_2)^4 m^{-8}$$

$$\sum_{h\in\mathbf{o}^2} \exp\{-2\pi m^{-1}((U\mathbf{h})({}^t\overline{\mathbf{h}}{}^t\overline{U}), \Lambda)$$

$$-2\pi T({}^t\overline{\mathbf{h}}{}^t\overline{U})({}^t\overline{U}^{-1})(\frac{\mathbf{q}}{m}-iV)\} \quad \text{(By 4-4)}$$

$$= e^{-2\pi m Y^{-1}[V]}(\det Y)^4 m^{-8} \sum_{h\in\mathbf{o}^2} \exp\{-2\pi m^{-1}(\mathbf{h}^t\overline{\mathbf{h}}, Y)$$

$$-2\pi i\,\sigma(\frac{q_1}{m}-iv_1, h_1) - 2\pi i\,\sigma(\frac{q_2}{m}-iv_2, h_2)\}$$

$$= e^{-2\pi m Y^{-1}[V]}(\det Y)^4 m^{-8}$$

$$\sum_{\mathbf{p}:(\mathbf{o}/m\mathbf{o})^2} \exp\{-2\pi i[\sigma(q_1,p_1)+\sigma(q_2,p_2)]/m\}$$

$$\sum_{\mathbf{h}=\lambda+\mathbf{p}/m,\ \lambda\in\mathbf{o}^2} \exp\{-2\pi m(\mathbf{h}^t\overline{\mathbf{h}}, Y)$$

$$-2\pi m(\sigma(v_1,h_1)+\sigma(v_2,h_2))\}$$

(Set $\mathbf{h}=\mathbf{p}+m\lambda$, $\mathbf{p}:(\mathbf{o}/m\mathbf{o})^2$, $\lambda:\mathbf{o}^2$)

$$= e^{-2\pi m Y^{-1}[V]}(\det Y)^4 m^{-8}$$

$$\sum_{\mathbf{p}:(\mathbf{o}/m\mathbf{o})^2} \exp\{-2\pi i[\sigma(q_1,p_1)+\sigma(q_2,p_2)]/m\}$$

$$\vartheta_{m,\mathbf{p}}(iY, iV).$$

This proves our assertion. □

We then have a result similar to Proposition 5.

Proposition 12. *There exists a group homomorphism ψ_2: $\Gamma_2 \to U(m^{16})$ (unitary group of size m^{16}) determined by*

(1) $\psi_2(p_B)=\mathrm{diag}\,[e^{-2\pi i(\mathbf{q}^t\overline{\mathbf{q}},B)}]_{\mathbf{q}:(\mathbf{o}/m\mathbf{o})^2}$, $B\in\Lambda_2$,

(2) $\psi_2(t_U)=[s_{\mathbf{p},\mathbf{q}}]_{\mathbf{p},\mathbf{q}:(\mathbf{o}/m\mathbf{o})^2}$, $s_{\mathbf{p},\mathbf{q}}=\begin{cases}1 & \text{if } \mathbf{q}=U\mathbf{p},\\ 0 & \textit{otherwise},\end{cases}$ *for* $U=\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}$ *or* $\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$, $t\in\mathbf{o}$,

(3) $\psi_2(\iota)=\frac{1}{m^8}[e^{2\pi i[\sigma(p_1,q_1)+\sigma(p_2,q_2)]/m}]_{\mathbf{p},\mathbf{q}:(\mathbf{o}/m\mathbf{o})^2}$.

VIII. Jacobi-Eisenstein Series

As shown in Proposition 10, we are able to decompose a Jacobi form of degree two into an inner product of a vector-valued modular form and a vector-valued theta series. Now, with the properties **(27)**, **(28)** and **(29)** of the theta series defined in **(26)**, we can characterize a Jacobi form as a vector-valued modular form.

Proposition 13. *Let $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_{m^{16}}$ be a set of representatives of $(\mathbf{o}/m\mathbf{o})^2$ and*

$$F(Z)={}^t(F_{\mathbf{q}_1}(Z), F_{\mathbf{q}_2}(Z), \ldots, F_{\mathbf{q}_{m^{16}}}(Z)) \tag{31}$$

$$\Theta(Z,W)={}^t(\vartheta_{m,\mathbf{q}_1}(Z,W), \ldots, \vartheta_{m,\mathbf{q}_{m^{16}}}(Z,W)) \tag{32}$$

*with $\vartheta_{m,\mathbf{q}}(Z,W)$ as defined in **(26)** and*

$$F_{\mathbf{q}}(Z)=\sum_{T\in\Lambda_2,\ T\geq \mathbf{q}^t\overline{\mathbf{q}}/m} \alpha(T,\mathbf{q})e^{2\pi i(T-\mathbf{q}^t\overline{\mathbf{q}}/m,\,Z)}.$$

Then, the following statements are equivalent:

(1) *$f(Z,W)={}^tF(Z)\bullet\Theta(Z,W)$ is a Jacobi form of wieght k and index m with respect to Γ_2.*

(2) *$F(Z)$ satisfies the following conditions:*

(i) $F(Z+B)=\psi_2(p_B)F(Z)$ *for* $B\in\Lambda_2$,

(ii) $F(Z[U])=\psi_2(t_U)F(Z)$ for $U=\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}$ or $\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$, $t\in\mathbf{o}$,

(iii) $F(-Z^{-1})=(\det Z)^{k-4}\psi_2(\iota)F(Z)$.

Proof. It is similar to the proof of Proposition 7, so we will omit it here. □

Corollary. *For a positive integer $k\geq 4$, the correspondence*

$$F(Z)\to F(Z)\,\vartheta_{1,\mathbf{0}}(Z,W)$$

establishes an one to one correspondence between modular forms of weight $k-4$ on $\mathcal{H}_2$ and Jacobi forms of weight k and index 1 on $\mathcal{H}_2\times C^2_{\mathbb{C}}$.

Now, we will use the group homomorphism ψ_2

to construct a vector-valued modular form corresponding to a Jacobi form of degree two. Let $j(g,Z)$ be a factor of the determinant of the Jacobian matrix of $g\in\Gamma_2$ at $Z\in\mathcal{H}_2$ determined by the following:

(1) $j(p_B,Z)=1$ for all $B\in\Lambda_2$,

(2) $j(t_U,Z)=1$ for all $U=\begin{bmatrix} 1 & t \\ 0 & 1\end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}$, $t\in\mathbf{o}$,

(3) $j(\iota,Z)=\det(-Z)$,

(4) $j(g_1g_2,Z)=j(g_1,g_2(Z))j(g_2,Z)$.

Also, let Γ_2^0 be the subgroup of Γ_2 generated by p_B and t_U, $B\in\Lambda_2$, $U=\begin{bmatrix} 1 & t \\ 0 & 1\end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}$, $t\in\mathbf{o}$. For each $\mathbf{q}\in\mathbf{o}^2$ with $\mathbf{q}^t\overline{\mathbf{q}}\equiv 0 \pmod m$, we define

$$E_{k,m}(Z,W;\mathbf{q}) = {}^t(E_{\mathbf{q},\mathbf{q}_1}(Z),\ E_{\mathbf{q},\mathbf{q}_2}(Z),\ \ldots,\ E_{\mathbf{q},\mathbf{q}_{m^{16}}}(Z))\bullet\Theta(Z,W) \quad \textbf{(33)}$$

with

$$E_{\mathbf{q},\mathbf{p}}(Z)=\sum_{M:\,\Gamma_2/\Gamma_2^0} j(M,Z)^{4-k}\overline{\psi_{2,\mathbf{q},\mathbf{p}}(M)}, \quad \textbf{(34)}$$

where

$$\psi_2(M)=(\psi_{2,\mathbf{q},\mathbf{p}}(M))_{\mathbf{q},\mathbf{p}\cdot(\mathbf{o}/m\mathbf{o})^2}. \quad \textbf{(35)}$$

The series in **(34)** converges absolutely and uniformly on compact subsets of $\mathcal{H}_2$ if $k>22$. Here, we shall prove that the vector-valued modular form corresponding to $E_{k,m}(Z,W;\mathbf{q})$ satisfies condition (2) of Proposition 13. Consequently, $E_{k,m}(Z,W;\mathbf{q})$ is indeed a Jacobi form of weight k and index m for $k>22$ and $\mathbf{q}^t\overline{\mathbf{q}}\equiv 0$ (mod m).

Proposition 14. *For $k>22$ and $\mathbf{q}\in\mathbf{o}^2$ with $\mathbf{q}^t\overline{\mathbf{q}}\equiv 0$ (mod m), the Jacobi-Eisenstein defined in (33) and (34) is a Jacobi form of weight k and index m.*

Proof. Let $E(Z;\mathbf{q})$ be the vector-valued modular form corresponding to $E_{k,m}(Z,W;\mathbf{q})$. Then, ${}^tE(Z;\mathbf{q})$ is the $\mathbf{q}$-th row of the matrix

$$\sum_{M:\,\Gamma_2/\Gamma_2^0} j(M,Z)^{4-k}\overline{\psi_2(M)}.$$

Thus, condition (2) of Proposition 13 follows from the cocycle condition of $j(M,Z)$ and the properties of ψ_2. □

IX. Applications to Singular Modular Forms

In addition to the Jacobi-Eisenstein series constructed in the previous section, the Jacobi-Fourier coefficients of modular forms on the exceptional domain provide another kind of examples of Jacobi forms of degree two. Here, we shall determine explicitly the Fourier coefficients of modular forms of weight 4 and 8 on the exceptional domain.

Proposition 15. *Let $E_4(Z)$ be a modular form of weight 4 on the exceptional domain with the Fourier expansion*

$$E_4(Z)=\sum_{T\in\mathcal{J}_o\cap\mathfrak{R}} a(T)\,e^{2\pi i(T,Z)}. \quad \textbf{(36)}$$

Then, $a(T)=0$ unless rank $T\le 1$. If $a(0)=1$ is given, then, for rank $T=1$,

$$a(T)=240\sum_{d\mid\varepsilon(T)} d^3,$$

where $\varepsilon(T)$ is the largest integer d such that $d^{-1}T\in\mathcal{J}_o$.

Proof. Let

$$\varphi_0(Z_1)+\sum_{m=1}^{\infty}\varphi_m(Z_1,W)\,e^{2\pi imz_3}$$

be the Jacobi-Fourier expansion of $E_4(Z)$ with

$$Z=\begin{bmatrix} Z_1 & W \\ {}^t\overline{W} & z_3\end{bmatrix}.$$

Then, $\varphi_0(Z_1)$ is a modular form of weight 4 on $\mathcal{H}_2$; hence, it is a constant multiple of $f_4(Z_1)$ in Proposition 1. Note that $a(0)=1$; it follows that $\varphi_0(Z_1)=f_4(Z_1)$ and that $a\begin{pmatrix} T_1 & 0 \\ 0 & 0\end{pmatrix}$ is given by $240\sum_{d\mid\varepsilon(T_1)} d^3$ if $\det T_1=0$, $T_1\neq 0$.

On the other hand, $\varphi_m(Z_1,W)$ is a Jacobi form of weight 4 and index m on $\mathcal{H}_2\times C_{\mathbb{C}}^2$. By Proposition 13, we are able to decompose $\varphi_m(Z_1,W)$ into

$$\sum_{\mathbf{q}:\,(\mathbf{o}/m\mathbf{o})^2} F_{\mathbf{q}}(Z_1)\bullet\vartheta_{m,\mathbf{q}}(Z_1,W)$$

with

$$F_{\mathbf{q}}(Z_1)=\sum_{T\in\Lambda_2,\,T\ge\mathbf{q}^t\overline{\mathbf{q}}/m} a\begin{pmatrix} T_1 & \mathbf{q} \\ {}^t\overline{\mathbf{q}} & 0\end{pmatrix} e^{2\pi i(T-\mathbf{q}^t\overline{\mathbf{q}}/m,\,Z_1)}.$$

By Proposition 13, we know that

$$F(Z_1)={}^t(F_{\mathbf{q}}(Z_1))_{\mathbf{q}:(\mathbf{o}/m\mathbf{o})^2}$$

is a vector-valued modular form of weight 0. This forces $F_{\mathbf{q}}(Z_1)$ to be a constant; hence,

$$a\begin{pmatrix} T_1 & \mathbf{q} \\ {}^t\overline{\mathbf{q}} & m \end{pmatrix}=0$$

unless $T_1=\mathbf{q}{}^t\overline{\mathbf{q}}/m$. This proves that $a(T)=0$ unless rank $T\leq 1$. For $T\in\mathcal{J}_o$ with rank $T=1$, we are able to reduce T to $T_0=\mathrm{diag}\,[\varepsilon(T),0,0]$ by using a finite number of operations $T\to T[U]$, $U=U_{ij}(t)$, $i\neq j$, $t\in\mathbf{o}$. Thus,

$$a(T)=a(T_0)=240\sum_{d|\varepsilon(T)} d^3. \qquad \square$$

Next, we will give a relation among the Fourier coefficients of the modular form of weight 4.

Proposition 16. *For each positive integer m and* $\mathbf{q}={}^t(q_1, q_2)\in\mathbf{o}^2$, *let*

$$T=\left(\begin{bmatrix} \mathbf{q}{}^t\overline{\mathbf{q}}/m & \mathbf{q} \\ {}^t\overline{\mathbf{q}} & m \end{bmatrix}\right)$$

and

$$G_m(\mathbf{q})=\begin{cases} 240\sum\limits_{d|\varepsilon(T)} d^3 & \text{if } T\in\mathcal{J}_o, \\ 0 & \text{otherwise.} \end{cases} \tag{37}$$

Then,

$$G_m(\mathbf{q})=\frac{1}{m^8}\sum_{\mathbf{p}:(\mathbf{o}/m\mathbf{o})^2} e^{2\pi i[\sigma(q_1,p_1)+\sigma(q_2,p_2)]/m}G_m(\mathbf{p}). \tag{38}$$

Proof. It follows from the fact that $G={}^t(G_m(\mathbf{q}))_{\mathbf{q}:(\mathbf{o}/m\mathbf{o})^2}$ is the vector-valued modular form corresponding to $\varphi_m(Z_1,W)$, the m-th Jacobi-Fourier coefficient of $E_4(Z)$. Thus, G must satisfy condition (2) of Proposition 13. In particular, we have

$$G=\psi_2(\iota)G. \tag{39}$$

This is precisely the identity **(38)** in vector form. $\square$

Remark. Indentity **(38)** was proved in Eie[1,2] directly from the definition of $G_m(\mathbf{q})$, and this implies that

$$\varphi_m(Z_1,W)=\sum_{\mathbf{q}:(\mathbf{o}/m\mathbf{o})^2} G_m(\mathbf{q})\vartheta_{m,\mathbf{q}}(Z_1,W),\ (Z_1,W)\in\mathcal{H}_2\times C_{\mathbb{C}}^2$$

is a Jacobi form of weight 4 and index m. With $\varphi_m(Z_1,W)$ as the m-th coefficient, we are able to define a holomorphic function on the exceptional domain as

$$E(Z)=f_4(Z_1)+\sum_{m=1}^{\infty}\varphi_m(Z_1,W)e^{2\pi imz_3},$$

$$Z=\begin{bmatrix} Z_1 & W \\ {}^t\overline{W} & z_3 \end{bmatrix}\in\mathcal{H}.$$

With the theory of Jacobi forms on $\mathbf{H}_1\times C_{\mathbb{C}}$ as well as Jacobi forms on $\mathcal{H}_2\times C_{\mathbb{C}}$, we are able to verify that

$$E(-Z^{-1})=(\det Z)^4E(Z). \tag{40}$$

Consequently, we can provide another way to construct the singular modular form of weight 4 on the exceptional domain.

Note that $E_4^2(Z)$ is a modular form of weight 8. Indeed, it is a singular modular form of weight 8, and its Fourier coefficiens can be determined explicitly by the following proposition.

Proposition 17. *Let*

$$E_4^2(Z)=\sum_{T\in\mathcal{J}_o\cap\mathfrak{R}} b(T)e^{2\pi i(T,Z)}.$$

Then, $b(T)=0$ *unless rank* $T\leq 2$ *and*

$$b\left(\begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix}\right)=\begin{cases} 1 & \text{if } T_1=0, \\ 480\sum\limits_{d|\varepsilon(T_1)} d^7 & \text{if } \det T_1=0, T_1\neq 0, \\ 240\cdot 480\sum\limits_{d|\varepsilon(T_1)} d^7\sum\limits_{d|\det(d^{-1}T_1)} d_1^3 & \text{if } \det T_1\neq 0. \end{cases}$$

Proof. The Fourier coefficient $a(T)$ of $E_4(Z)$ has the property that $a(T)=0$ unless rank $T\leq 1$. It follows that

$$b(T)=\sum_{T_1+T_2=T} a(T_1)a(T_2)$$

is zero unless rank $T\leq 2$. Let

$$\psi_0(Z_1)+\sum_{m=1}^{\infty}\psi_m(Z_1,W)e^{2\pi imz_3}$$

be the Jacobi-Fourier expansion of $E_4^2(Z)$. Then,

[1]Eie, M., "The cohomology group associated with Jacobi cusp forms over Cayley numbers." To appear in *Amer. Jour. of Math.*
[2]Eie, M., "An arithmetic property of Fourier coefficients of singular modular forms on the exceptional domain." Manuscript.

$$\psi_0(Z_1)=\sum_{T_1\in\Lambda_2} b\left(\begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix}\right) e^{2\pi i(T_1,Z_1)}$$

$$=\lim_{\lambda\to\infty} E_4^2\left(\begin{bmatrix} Z_1 & 0 \\ 0 & i\lambda \end{bmatrix}\right),$$

which is a modular form of weight 8; hence, it is equal to $[f_4(Z_1)]^2$. However, $[f_4(Z_1)]^2$ is the only modular form of weight 8, which is also in the Maaβ space, and its coefficients satisfy the Maaβ condition. Therefore, it suffices to show that $b\left(\begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix}\right)$ with $T_1=\begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix}$ or $T_1=\begin{pmatrix} n & t \\ \bar{t} & 1 \end{pmatrix}$, $n-N(t)\neq 0$. Note that

$$b\left(\begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix}\right)=\#\{h_1,h_2\in \mathbf{o}^2 \mid h_2{}^t\bar{h}_1+h_2{}^t\bar{h}_2=T_1\}.$$

For $T_1=\begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix}$, we have

$$b\left(\begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix}\right)=\#\{a,b\in \mathbf{o} \mid N(a)+N(b)=n\}$$

$$=480\sum_{d\mid n} d^7.$$

On the other hand, for $T_1=\begin{bmatrix} n & t \\ \bar{t} & 1 \end{bmatrix}$, we have

$$b\left(\begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix}\right)=240\cdot 480\sum_{d\mid (n-N(t))} d^3.$$

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八元數上的二階Jacobi式理論

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摘　要

在這論文中，我們將描述作者於1991年所創的八元數上的一階Jacobi式，然後介紹二階Jacobi式理論。透過一組theta級數的轉換式所得到的一群表現，我們建構一族Eisenstein級數，是Jacobi式的另一類典型例子。