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Universal Scaling Functions for Percolation Models

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ABSTRACT

Many problems in mathematics, physical sciences, and life sciences can be described by percolation models. In this paper, we review our recent work in universal quantities and universal finite-size scaling functions (UFSSF's) of percolation models. The quantities we consider include the existence probability (also called the spanning probability), E_p, the percolation probability, P, and the probability of the appearance of n percolating clusters, W_n. The topics under discussion include: (1) boundary conditions, aspect ratios, and finite-size scaling functions; (2) UFSSF's of E_p and P in lattice percolation models; (3) UFSSF's of W_n in lattice percolation models; (4) UFSSF's of E_p and W_n in continuum percolation models; (5) UFSSF's of the q-state bond-correlated percolation model and q-state Potts model without nonuniversal metric factors; and (6) boundary conditions and the average number of percolating clusters. Some other related developments and problems for further research are also discussed.

Key Words: percolation, critical phenomena, finite-size scaling, universality

I. Introduction

Percolation has become an important branch of the sciences in recent decades and is related to many interesting mathematical and physical problems (Essam, 1973, 1980; Deutscher *et al*., 1983; Zallen, 1983; Stauffer and Aharony, 1994). In previous papers (Hu, 1988, 1990, 1992a, 1993), the author reviewed basic ideas of percolation and fractals and the usage of these ideas in the description of critical phenomena of Ising-type spin models and hard-core particle models based on the connection between these models and correlated percolation models (Hu, 1984a, 1984b; Hu and Mak, 1989a, 1990). In this paper, we will review our recent work in the Monte Carlo studies of universal finite-size scaling functions for percolation models (Hu, 1997, 1998; Hu *et al*., 1998, 1999a).

Universality and scaling are two important concepts in the theory of critical phenomena (Stanley, 1971; Kadanoff, 1990). The former dates from the work by Yang (1952) and Chang (1952). In 1952, Yang (1952) derived the exact spontaneous magnetization M of the Ising model on a square lattice with isotropic interactions and found that the critical exponent β of M is 1/8. In the same year, Chang (1952) derived the exact spontaneous magnetization M of the Ising model on a square lattice with anisotropic interactions; i.e., the coupling constants in the horizontal direction J_1 and in the vertical direction J_2 are different. Chang (1952) found that, for $0<J_1/J_2<\infty$, β is always equal to 1/8. Chang (1952) conjectured that for other planar lattices, β is also equal to 1/8, which was confirmed by later calculations: this marked the beginning of the theory of the universality of critical exponents. Now it is generally believed that for the Ising model on all planar lattices, including the square (sq), the plane triangular (pt), honeycomb (hc) lattices, etc., the specific heat exponent α, the spontaneous magnetization exponent β, the magnetic susceptibility exponent γ, and the correlation length exponent ν are 0 (logarithmic divergence), 1/8, 7/4, and 1, respectively (Stanley, 1971). It is also believed that for site and bond random percolation on all planar lattices, the correlation length exponent ν, the percolation probability exponent β, and the mean cluster size exponent γ are 4/3, 5/36, and 43/18, respectively (Stauffer and Aharony, 1994).

Another important concept in the theory of critical phenomena is scaling (Stanley, 1971; Kadanoff, 1990). For example, in a ferromagnetic system, e.g., $CrBr_3$, for temperatures T near the critical temperature T_c (also called the Curie temperature in ferromagnetic systems), if we plot $\sigma/|\epsilon|^{\beta}$ as a function of $h/|\epsilon|^{\beta+\gamma}$, where σ is the magnetization, $\epsilon=(T-T_c)/T_c$, and h is the external magnetic field, then the experimental data for different temperatures collapse on a single curve, called the scaling function (Stanley, 1971). In this paper, we consider another kind of scaling, called finite-size scaling.

According to the theory of finite-size scaling (Fisher, 1971; Barber, 1983; Privman and Fisher, 1984; Cardy, 1988; Privman, 1990; Stauffer and Aharony, 1994), if the dependence of a physical quantity Q of a thermodynamic system on the parameter ϵ, which vanishes at the critical point ϵ=0, is of the form $Q(\epsilon)\sim|\epsilon|^a$ near the critical point, then for a finite system of linear dimension L, the corresponding quantity $Q(L,\epsilon)$ is of the form

$$Q(L,\ \epsilon)\approx L^{-ay_t}F(\epsilon L^{y_t}), \tag{1}$$

where y_t ($=\nu^{-1}$) is the thermal scaling power and $F(x)$ is the finite-size scaling function. It follows from Eq. (1) that the scaled data $Q(L,\epsilon)L^{ay_t}$ for different values of L and ϵ can be described as a single function of the scaling variable $x=\epsilon L^{y_t}$. Thus it is important to know general features of the finite-size scaling function under various conditions.

In 1984, Privman and Fisher (1984) proposed the idea of universal finite-size scaling functions (UFSSF's) and that of nonuniversal metric factors for static critical phenomena (Privman and Fisher, 1984) for T near T_c and the external magnetic field h near 0. Specifically, they proposed that, near ϵ=0 and h=0, the singular part of the free energy for a ferromagnetic system can be written as

$$f_s(\epsilon,h,L)\approx L^{-d}Y(C_1\epsilon L^{1/\nu},\ C_2hL^{(\beta+\gamma)/\nu}), \tag{2}$$

where d is the spatial dimensionality of the lattice, Y is a universal finite-size scaling function, and C_1 and C_2 are adjustable nonuniversal metric factors (Privman and Fisher, 1984), which depend on the specific lattice structure. From Eq. (2) and the scaling relations νd= $2-\alpha$ and $\alpha+2\beta+\gamma$=2 (Stanley, 1971), one can obtain the scaling expression for the finite-size magnetization (Privman and Fisher, 1984)

$$m=-\frac{\partial}{\partial h}f_s(\epsilon,h,L)$$

$$\approx C_2L^{-\beta/\nu}Y^{(1)}(C_1\epsilon L^{1/\nu},C_2hL^{(\beta+\gamma)/\nu}), \tag{3}$$

which is the order parameter of the system. From 1984 to 1994, progress in research on UFSSF's was very slow. The title of Privman and Fisher's paper (Privman and Fisher, 1984) is "Universal critical amplitudes in finite-size scaling," and most papers related to this paper only address the problem of the universality of critical amplitudes rather than that of the universality of finite-size scaling functions.

In 1992, Hu (1992b, 1992c) proposed a histogram Monte Carlo simulation method (HMCSM) to study random and correlated percolation models. In 1995~1996, Hu and his cooperators applied the HMCSM to calculate the existence probability (also called the spanning probability, see Langlands *et al*. (1992)) E_p, the percolation probability P, and the probability W_n for the appearance of n top to bottom percolating clusters on finite square (sq), honeycomb (hc), and planar triangular (pt) lattices. They found that, by choosing an appropriate aspect ratio for each lattice and nonuniversal metric factors for each model, E_p, P, and W_n for six percolation models on planar lattices have UFSSF's (Hu *et al*., 1995a, 1995b; Hu and Lin, 1996). In 1997, Hu and Wang (1997) used a random deposition Monte Carlo method to find that the continuum percolation of soft disks and hard disks have the same UFSSF's as does percolation on planar lattices. Based on the connection between the q-state Potts model (QPM) (Wu, 1982) and a q-state bond-correlated percolation model (QBCPM) (Hu, 1984a, 1984b, 1992a), Hu *et al.* (1999a, 1999b) used the HMCSM and a cluster Monte Carlo simulation method (Swendsen and Wang, 1987) to calculate the UFSSF's for the QPM on sq, hc, and pt lattices. When an appropriate scaling variable for the QPM is used, they can obtain UFSSF's without using any adjustable parameter. Very recently, we used the HMCSM to study the relation between boundary conditions and the average number of percolating clusters, C. We found that for lattices with four different boundary conditions and large aspect ratios, C increases linearly with R with a slope which is independent of the boundary conditions. In this paper, we briefly review the above developments in Monte Carlo approaches to UFSSF's in percolation models (see also Hu (1997, 1998) and Hu *et al*. (1998, 1999b)).

This paper is organized as follows. In Sec. II we review the histogram Monte Carlo simulation method (Hu, 1992b) and the use of this method to calculate finite-size scaling functions, critical points, critical exponents, and thermodynamic order parameters for percolation on lattices under various conditions. In Sec. III we review the histogram Monte Carlo approach to the UFSSF's for the existence probability E_p and the percolation probability P in lattice percolation models. In Sec. IV we review the histogram Monte Carlo approaches to UFSSF's for the probability of the appearance of n percolating clusters, W_n, in lattice percolation models. In Sec. V we present our Monte Carlo results for UFSSF's in continuum percolation of soft disks and hard disks. In Sec. VI we present our Monte Carlo results for UFSSF's of QBCPM and QPM without nonuniversal metric factors. In Sec. VII we present our Monte Carlo results for the average number of percolating clusters under various boundary conditions. Some other related developments are mentioned in Sec. VIII, and problems for further research are also discussed in Sec. IX.

II. Histogram Monte Carlo Simulation Method and Its Applications

In 1992, Hu (1992b, 1992c) proposed HMCSM, which was then used to calculate the finite-size scaling functions for the existence probability E_p and the percolation probability P of the percolation model and the q-state bond-correlated percolation model corresponding to the q-state Potts model (Chen and Hu, 1993; Hu, 1992d, 1994a, 1994b; Hu and Chen, 1993, 1995; Hu *et al*., 1996). Here E_p is the probability that the system percolates, and P is the probability that a given lattice site belongs to a percolating cluster. Hu and his cooperators found that E_p and P have very good finite-size scaling behavior, and that finite-size scaling functions depend sensitively on the boundary conditions and aspect ratio of the lattice and on spanning rules to define percolating clusters (Hu, 1994a, 1994b; Hu and Chen, 1995; Hu *et al*., 1996).

Now we will briefly review the HMCSM for bond percolation (Hu, 1992b, 1992c; Hu *et al*., 1996) on an $L_1 \times L_2$ square lattice G and define related quantities, where L_1 is the linear dimension in the horizontal direction and L_2 is the linear dimension in the vertical direction. The extension to other lattices and to site percolation (Hu, 1994a, 1994b) is straightforward. In this paper, we consider the following four possible boundary conditions of the $L_1 \times L_2$ lattice:

- BC1. periodic in the L_1 direction and free in the L_2 direction.
- BC2. free in both the L_1 and L_2 directions.
- BC3. periodic in both the L_1 and L_2 directions.
- BC4. free in the L_1 direction and periodic in the L_2 direction.

In bond percolation on a lattice G with N sites, $N=L_1 \times L_2$, and E bonds, each bond of G is occupied by a probability p, where $0 \le p \le 1$. There are several different rules used to define percolating clusters, called spanning rules, which were first discussed by Reynolds *et al*. (1980). In R_1, a cluster percolates if it extends from the top row of G to the bottom row of G; in R_2, a cluster percolates if it extends from the top row to the bottom row and from the left boundary to the right boundary of G (Reynolds *et al*., 1980). In a given spanning rule, a subgraph which contains at least one percolating cluster is a percolating subgraph and is denoted by G'_p. In a previous paper (Hu, 1994b), I used both R_1 and R_2 to define percolating clusters. In the following, I will only use R_1 to define percolating clusters. Then, we have the following definitions for the existence probability $E_p(G,p)$ and the percolation probability $P(G,p)$:

$$E_p(G,p) = \sum_{G'_p \subseteq G} p^{b(G'_p)}(1-p)^{E-b(G'_p)}, \tag{4}$$

$$P(G,p) = \sum_{G'_p \subseteq G} p^{b(G'_p)}(1-p)^{E-b(G'_p)} N^*(G'_p)/N, \tag{5}$$

where $b(G'_p)$ is the number of occupied bonds in G'_p and $N^*(G'_p)$ is the total number of sites in the percolating clusters of G'_p. The summations in Eqs. (4) and (5) are over all subgraphs G'_p of G.

To carry out histogram Monte Carlo simulations, we first choose w different values of p. For a given $p=p_j$, $1 \le j \le w$, we generate N_R different subgraphs G'. The data obtained from the wN_R different G' are then used to construct three arrays of numbers of length E with elements $N_p(b)$, $N_f(b)$, and $N_{pp}(b)$, which are, respectively, the total numbers of percolating subgraphs with b occupied bonds, nonpercolating subgraphs with b occupied bonds, and the sum of $N^*(G'_p)$ over subgraphs with b occupied bonds. After a sufficient number of simulations, these arrays can be used to obtain approximate E_p and P for any value of the bond occupation probability p (Hu, 1992b; Hu *et al*., 1996):

$$E_p(G,p) = \sum_{b=0}^{E} p^b(1-p)^{E-b} C_b^E \frac{N_p(b)}{N_p(b)+N_f(b)}, \tag{6}$$

$$P(G,p) = \frac{1}{N}\sum_{b=0}^{E} p^b(1-p)^{E-b} C_b^E \frac{N_{pp}(b)}{N_p(b)+N_f(b)}, \tag{7}$$

where $C_b^E = E!/(E-b)!b!$. Once we have histogram data, we can calculate E_p and P as continuous functions of p. This is different from the traditional Monte Carlo methods (Binder, 1986; Stauffer and Aharony, 1994).

Suppose we have already carried out histogram Monte Carlo simulations on lattices G_1 and G_2 of linear dimensions L_i and L_f, respectively, where $L_i > L_f$. The percolation renormalization group (PRG) transformation from lattice G_1 to lattice G_2 is given by the equation (Hu, 1992b, 1992c; Hu and Chen, 1988a, 1988b)

$$E_p(G_2,p') = E_p(G_1,p), \tag{8}$$

which gives the renormalized bond probability p' as a function of p. The fixed point of Eq. (8) gives the critical point p_c, i.e.,

$$E_p(G_2,p_c) = E_p(G_1,p_c). \tag{9}$$

The thermal scaling power y_t and the field scaling power y_h, which is equal to the fractal dimension D of the percolating cluster at p_c (Stanley, 1977; Hu, 1992d), may be obtained from the following equations:

$$\frac{1}{\nu}=y_t=\frac{\ln(\frac{\partial p'}{\partial p})_{p_c}}{\ln\frac{L_i}{L_f}}\,,\quad y_h=D=\frac{\ln\frac{P(G_i',p_c)L_i^d}{P(G_f',p_c)L_f^d}}{\ln\frac{L_i}{L_f}}\,. \quad \mathbf{(10)}$$

We may associate with each site of the lattice an adimensional "magnetic moment" m_0 and consider the renormalization of m_0 under the PRG transformation to give the renormalized "magnetic moment" m_0' (Hu and Chen, 1989; Tsallis *et al.*, 1985):

$$m_0'P(G_2,p')L_2^d=m_0P(G_1,p)L_1^d\,, \quad \mathbf{(11)}$$

which means that the total "magnetization" is preserved after the PRG transformation. After a series of PRG transformations, we have a series of renormalized bond probabilities p, $p^{(1)}(=p')$, $p^{(2)}$, ..., $p^{(n)}$ and the renormalized magnetic moments m_0, $m_0^{(1)}(=m_0')$, $m_0^{(2)}$, ..., $m_0^{(n)}$. The thermodynamic percolation probability of the original system, $P_\infty(p)$, may be related to the thermodynamic percolation probability of the n-th transformed system, $P_\infty(p^{(n)})$, by the following equation:

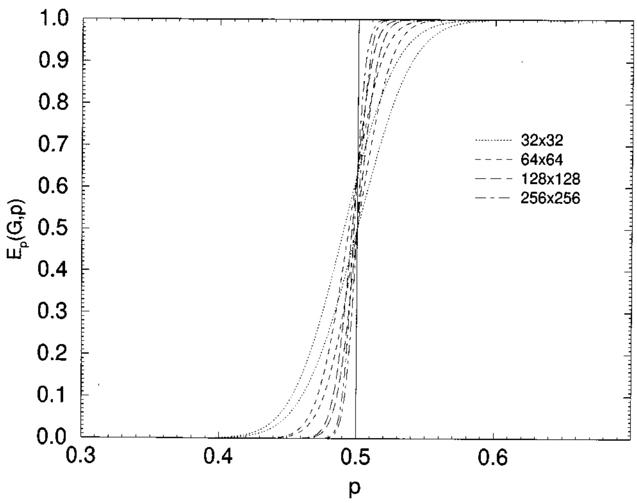
$$P_\infty(p)=\frac{m_0^{(n)}}{\lambda^{nd}m_0}P_\infty(p^{(n)})\,, \quad \mathbf{(12)}$$

for $p>p_c$ with $\lambda=L_i/L_f$. In the traditional small cell renormalization group transformation (RGT) (Tsallis *et al.*, 1985; Hu and Chen, 1989), one iterates the RGT's until $p^{(n)}$ approaches the "lower temperature" fixed point $p_c=1$; then $P_\infty(p^{(n)})$ of Eq. (12) is given by 1. However, in the large cell-to-cell RGT's considered here, one needs to only iterate the RGT's until the correlation length of the n-th transformed system is smaller than the linear dimensions of the transformed cell (Hu, 1995). In this case, the transformed cell may well represent the thermodynamic systems, and we may use $P(G_2,p^{(n)})$ to represent $P_\infty(p^{(n)})$ of Eq. (12) and obtain (Hu, 1994a, 1995)

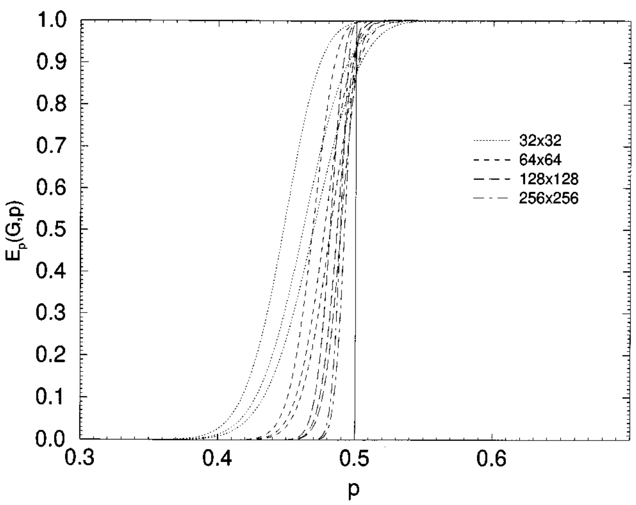
$$P_\infty(p)=\frac{m_0^{(n)}}{\lambda^{nd}m_0}P(G,p^{(n)})\,. \quad \mathbf{(13)}$$

Now we can begin to use the above equations to calculate E_p, P, and their finite-size scaling functions. Typical calculated results of E_p for bond percolation on $L_1{\times}L_2$ square lattices with BC1 to BC4 are shown in Fig. 1(a) and (b). Typical calculated results of P for bond percolation on $L{\times}L$ square lattices with BC1 are shown in Fig. 2. As $L=\sqrt{L_1L_2}\to\infty$, E_p is 0 for $p<p_c$ and is 1 for $p>p_c$; if we write $E_p\sim(p-p_c)^a$ just above p_c, then the critical exponent a of E_p is 0 (Stauffer and Aharony, 1994). On the other hand, $P\sim(p-p_c)^\beta$ just above p_c. According to Eq. (1), we may write $E_p=F(x)$ and $PL^{\beta/\nu}=S(x)$ with $x=(p-p_c)L^{1/\nu}$, where $F(x)$ and $S(x)$ are scaling functions.

For bond percolation on a square lattice, it is generally believed that the *exact* ν, β and p_c are 4/3, 5/36, and 1/2, respectively (Stauffer and Aharony, 1994). Using the exact values of ν and p_c, we have plotted



(a)



(b)

Fig. 1. The calculated $E_p(G,p)$ as a function of p for the bond random percolation model on $L_1{\times}L_2$ square lattices with different boundary conditions. The vertical line intersects the p axis at p_c=0.5. (a) BC1 and BC2 with $R=L_1/L_2$=1. (b) BC3 and BC4 with R=1 and 2.

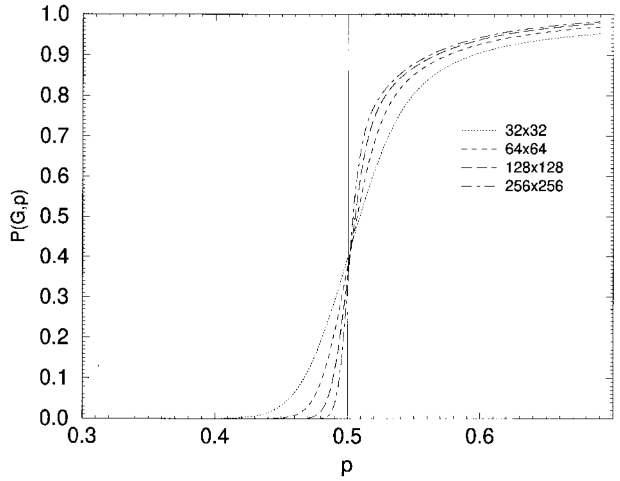


Fig. 2. The calculated $P(G,p)$ as a function of p for the bond random percolation model on $L \times L$ square lattices with BC1.

the data for E_p represented in Fig. 1(a) and (b) as a function of $x=(p-p_c)L^{1/\nu}$ in Fig. 3(a) and (b), respectively. Using the same values of ν and p_c, we have also plotted $PL^{\beta/\nu}$ for P presented in Fig. 2 as a function of $x=(p-p_c)L^{1/\nu}$ in Fig. 4. Figures 3 and 4 show that E_p and P have nice finite-size scaling behavior. Figure 3(a) and (b) show that scaling functions for different boundary conditions and aspect ratios are quite different.

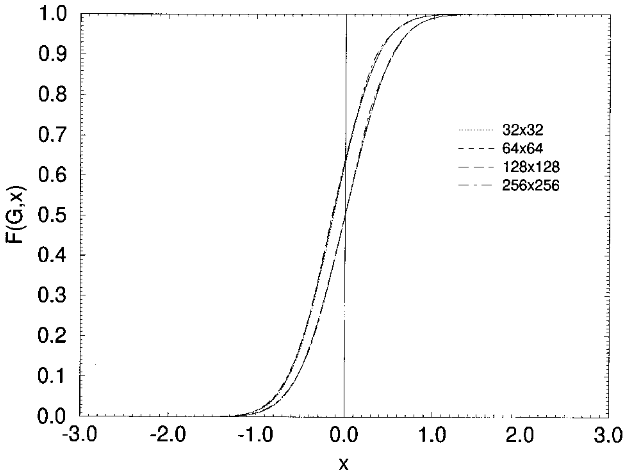
To show the reliability of our method, we have used the above percolation renormalization group equations (Hu, 1992b, 1994a, 1995) to calculate the critical point p_c, the thermal scaling power $y_t(=1/\nu)$ and the field scaling power $y_h=D=d-\beta/\nu$ for site and bond percolation on two and three dimensional lattices with different boundary conditions and aspect ratios. The calculated y_t, D, and p_c for bond percolation on a square lattice are very close to the exact results (Hu, 1992b; Hu *et al.*, 1996). The calculated p_c for site percolation on a square lattice is very close to other numerical results (Hu, 1994a). The calculated y_t, D, and p_c for site and bond percolation on three-dimensional lattices is very close to other numerical results (Lin *et al.*, 1998; Lin and Hu, 1998).

We have used Eq. (13) to calculate the thermodynamic order parameter P_∞ for site and bond percolation on square lattices with different boundary conditions, spanning rules, and aspect ratios. Although scaling functions strongly depend on these factors, the calculated thermodynamic order parameter P_∞ is independent of these factors (Hu, 1994a, 1994b; Hu *et al.*, 1996).

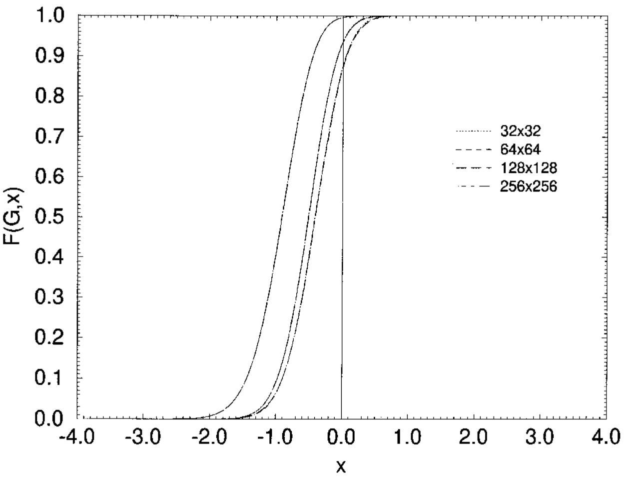
In summary, we have found that E_p and $PL^{\beta/\nu}$ have very good scaling behavior, and that the finite-size scaling functions depend sensitively on boundary conditions and aspect ratios of the lattice. However, the calculated ν, D, p_c, and P_∞ are independent of the boundary conditions and aspect ratios of the lattice.

III. UFSSF's for E_p and P of Lattice Percolation Models

Equation (1) for E_p implies that E_p for all models in the same universality class must be equal at the critical point in order to have UFSSF's. In 1992, Ziff (1992) found that E_p=0.5 for site and bond percolation on large square lattices with free boundary conditions, and Langlands *et al.* (1992) proposed that when aspect



(a)



(b)

Fig. 3. The data of E_p in Fig. 1 as a function of x, where $x=(p-p_c)L^{y_t}$. The function is the scaling function $F(G,x)$. (a) BC1 (bottom curves) and BC2 (top curves). (b) BC3 with R=2 (top curves), BC3 with R=1 (middle curves), and BC4 with R=1 (bottom curves).

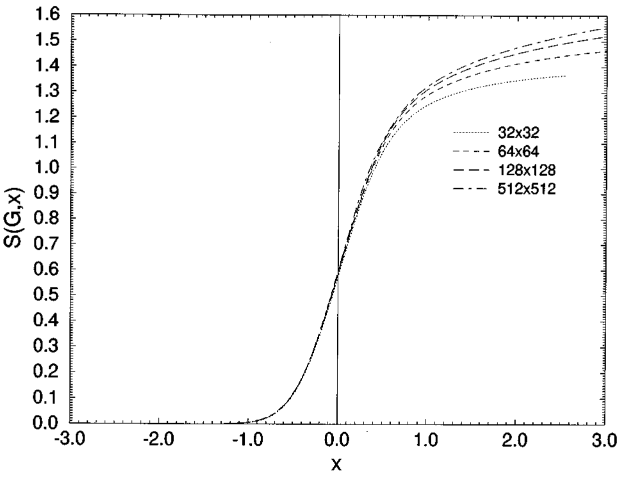
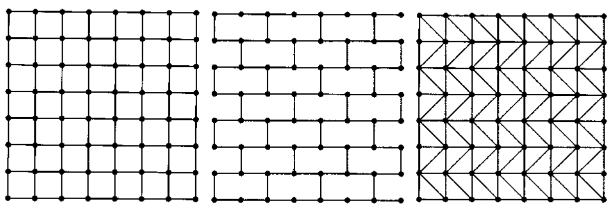


Fig. 4. The calculated $P/L^{-\beta y_t}$ in Fig. 2 as a function of x, where $x=(p-p_c)L^{-y_t}$. The function is the scaling function $S(G,x)$.



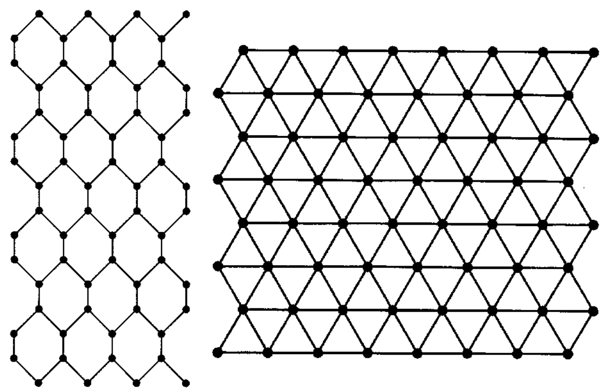
(a) (b) (c)

Fig. 5. (a) A 8×8 square (sq) lattice. (b) A 8×8 honeycomb (hc) lattice obtained from a 8×8 sq lattice by deleting half of the vertical bonds. (c) A 8×8 plane triangular (pt) lattice obtained from a 8×8 sq lattice by adding diagonal bonds.

ratios for sq, hc, and pt lattices have the relative proportions $1{:}\sqrt{3}{:}\sqrt{3}/2$, then site and bond percolation on such lattices have the same value of E_p at the critical point (Langlands *et al.*, 1992). In 1992, Cardy used a conformal theory to write a formula for the critical E_p as a function of the aspect ratio for percolation on lattices with free boundary conditions (Cardy, 1992). Cardy's formula (Cardy, 1992) is consistent with numerical results of Langlands *et al.* (1992) . Cardy (1992) and Langlands *et al.* (1992) did not discuss the values of E_p for $p \neq p_c$.

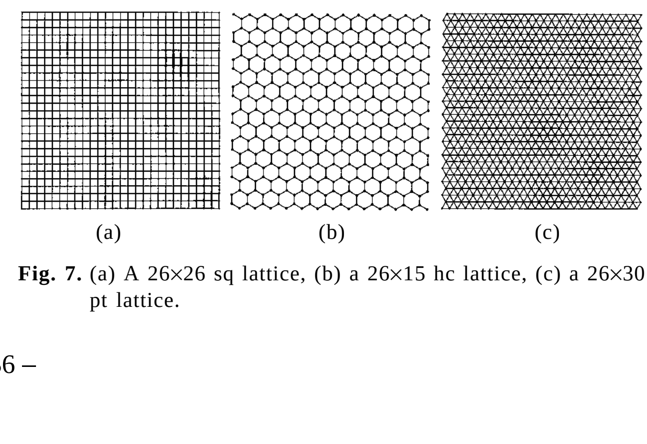
Result of Langlands *et al.* (1992) for the relative proportions $1{:}\sqrt{3}{:}\sqrt{3}/2$ for sq, hc, and pt lattices can be understood as follows. Figure 5(a) shows a typical $L\times L$ sq lattice. An $L\times L$ hc and an $L\times L$ pt lattices can be obtained from the sq lattice of Fig. 5(a) by removing or adding bonds, respectively, as shown in Fig. 5(b) and (c), which are equivalent to Fig. 6(a) and (b), respectively. If lattice sites in the horizontal direction of Fig. 6(a) and (b) are enlarged by a factor $\sqrt{3}$ and $\sqrt{3}/2$, respectively, the domains of hc and pt lattices are similar to $L\times L$ square lattices. To illustrate this point, we show 26×26 sq, 26×15 hc, and 26×30 pt lattices in Fig. 7(a)-(c), whose aspect ratios approximately match the ratio $1{:}\sqrt{3}{:}\sqrt{3}/2$ considered by Langlands *et al.* (1992). It is obvious that three figures in Fig. 7 have similar domains.

Hu *et al.* (1995a) applied the HMCSM (Hu, 1992b) to calculate E_p and P of site and bond percolation on finite 512×512 sq, 433×250 hc, and 433×500 pt lattices; i.e., they used 512/512:433/250:433/500 to approximate the proportions $1{:}\sqrt{3}{:}\sqrt{3}/2$ of aspect ratios for sq, hc, and pt lattices considered by Langlands *et al.* (1992). The results for E_p and P are reproduced in Fig. 8(a) and (b), respectively. Plotting E_p as a function of $x=D_1(p-p_c)L^{1/\nu}$ and $D_3PL^{\beta/\nu}$ as a function of $x=D_2(p-p_c)L^{1/\nu}$, where D_1, D_2 and D_3 are nonuniversal metric factors, Hu *et al.* (1995a) found that the six percolation models have very nice universal finite-size scaling functions for E_p and P, which are reproduced in Fig. 9(a) and (b), respectively. Within numerical uncertainties, $D_1=D_2$ and the nonuniversal metric factors for periodic boundary conditions are consistent with those for free boundary conditions although the scaling functions are quite different. Hu *et al.* (1995a, 1995b) also found that the nonuniversal metric factors are



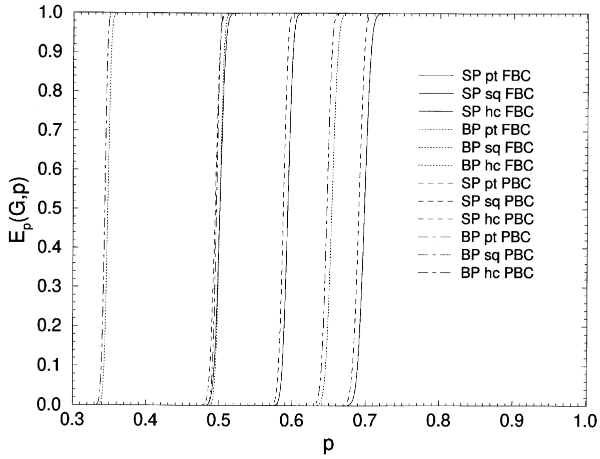
(a) (b)

Fig. 6. (a) An $L\times L$ hc lattice with aspect ratio $a=1$ and $L=8$. (b) An $L\times L$ pt lattice with aspect ratio $a=1$ and $L=8$.

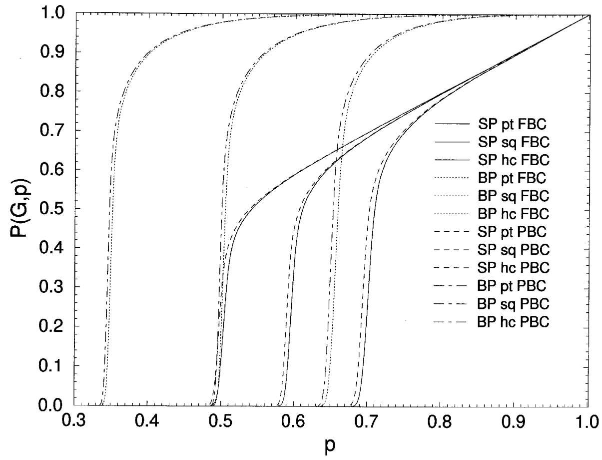


(a) (b) (c)

Fig. 7. (a) A 26×26 sq lattice, (b) a 26×15 hc lattice, (c) a 26×30 pt lattice.



(a)

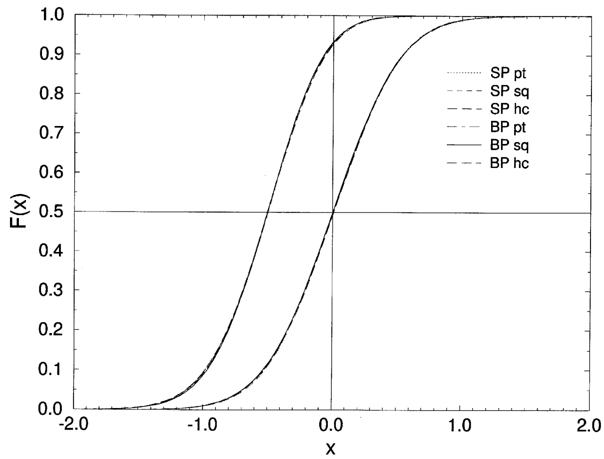


(b)

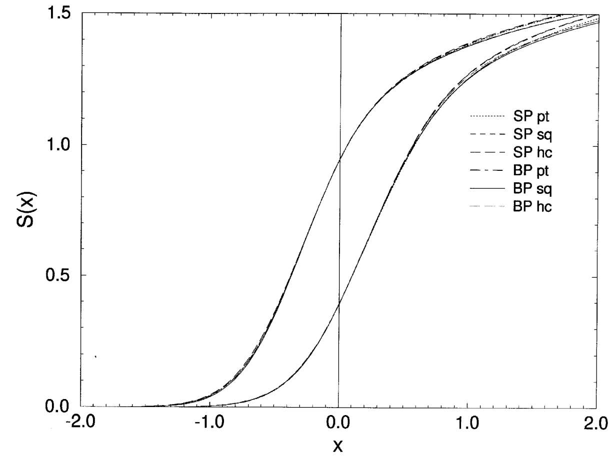
Fig. 8. Results for site percolation (SP) and bond percolation (BP) on pt, sq, and hc lattices. The solid (dotted) lines from left to right are for site (bond) percolation on pt, sq, and hc lattices with free boundary conditions (FBC). The dashed (dot-dashed) lines from left to right are for site (bond) percolation on pt, sq, and hc lattices with periodic boundary conditions (PBC). (a) E_p as a function of p. (b) P as a function of p.

independent of changes in aspect ratios holding the ratio between them constant (Hu *et al*., 1995b). These results indicate for each percolation model, we need only two nonuniversal metric factors, i.e., D_1 and D_3.

After Hu *et al*. (1995a) was published, Okabe and Kikuchi (1996) obtained universal finite-size scaling functions for the two-dimensional Ising model, and Hovi and Aharony (1996) calculated the scaling function $f(x)$ for bond and site percolation on the square lattice with both free (f) and periodic (p) boundary conditions (bc). Hovi and Aharony (1996) found that their $f(x)$ for fbc is consistent with $f(x)$ of Hu *et al*. (1995a), but their $f(x)$ for pbc is quite different from $f(x)$; i.e., Hovi and Aharony (1996) obtained $f(0)=0.63665\pm0.0008$, and Hu *et al*. (1995a, 1995b) obtained $f(0)=0.93(4)$. Hu (1996a) conjectured that the difference was because Hovi and Aharony (1996) considered pbc only in one direction while Hu *et al*. (1995a, 1995b) considered pbc in both the horizontal and vertical directions. This conjecture was confirmed by numerical calculations (Hu, 1996a). This result provided further evidence that finite-size scaling functions sensitively depend on the boundary conditions (Hu, 1994a).



(a)



(b)

Fig. 9. (a) The calculated E_p for site and bond percolation on pt, sq, and hc lattices as a function of x, where $x=D_1(p-p_c)L^{y_t}$. The scaling function is $F(x)$. The lower (upper) curves are for free (periodic) boundary conditions. (b) The calculated $D_3P/L^{-\beta y_t}$ for site and bond percolation on pt, sq, and hc lattices as a function of x, where $x=D_2(p-p_c)L^{y_t}$. The scaling function is $S(x)$. The lower (upper) curves are for free (periodic) boundary conditions.

IV. UFSSF for W_n of Lattice Percolation Models

In low-temperature measurements of quantum Hall effects (QHE), when the external magnetic field is increased from small values to large values, the conductivity σ_{xy} moves from one plateau with $\sigma_{xy}=\sigma_1$ to another plateau with the value $\sigma_{xy}=\sigma_2$, and the conductivity σ_{xx} has a maximum $\sigma_{xx}^{\max}$ in the transition region. In a recent theory of QHE, Ruzin *et al.* (1996) proposed that the number of percolating clusters in the sample at the critical point is useful for understanding $\sigma_{xx}^{\max}$. Therefore, the number of percolating clusters in percolation problems is an interesting quantity, and we should know more about its behavior.

To mimic the Corbino disk often used in experimental studies of quantum Hall effects (Ruzin *et al.*, 1996), Hu (1996b) used the HMCSM to study bond percolation on $L_1 \times L_2$ square lattices G with pbc in the horizontal L_1 direction and fbc in the vertical L_2 direction. Here "square" means that the primitive unit cell of the lattice is a square. A cluster which extends from the top row of G to the bottom row of G is a percolating cluster. A subgraph which contains at least one percolating cluster is a percolating subgraph and is denoted by G_p'. It should be noted that the definition of G_p' in Hu (1996b) and Hu and Lin (1996) and in this section is different from that of Hu (1994a, 1994b), Hu and Chen (1995) and Hu *et al*. (1995a, 1995b), in which only the largest cluster is used to define G_p'. A percolating subgraph which contains exactly n percolating clusters is denoted by G_n'. Now we have the definition

$$W_n(L_1, L_2, p) = \sum_{G_n' \subseteq G} p^{b(G_n')}(1-p)^{E-b(G_n')}, \qquad \mathbf{(14)}$$

where $b(G_n')$ is the number of occupied bonds in G_n'. The summation in Eq. (14) is over all subgraphs G_n' of G. To use the HMCSM to evaluate W_n, in addition to $N_p(b)$ and $N_f(b)$ considered in Sec. II, we also evaluate $N_n(b)$, $0 \le b \le E$, which is the number of percolating subgraphs with b occupied bonds and n percolating clusters. After a large number of simulations, the probability $W_n(L_1,L_2,p)$ at any value of the bond occupation probability p can be calculated approximately using the following equation (Hu, 1992b, 1996b):

$$W_n(L_1, L_2, p) = \sum_{b=0}^{E} p^b (1-p)^{E-b} C_b^E \frac{N_n(b)}{N_p(b) + N_f(b)} . \qquad \mathbf{(15)}$$

It is obvious that $E_p = \sum_{n=1}^{\infty} W_n$ and $W_0(L_1,L_2,p)=1-E_p$. Hu (1996b) found that W_n as a function of $z=(p-p_c)L^{1/\nu}$ has very good scaling behavior. Hu (1996b) also considered fbc in both the horizontal and vertical directions and found that the scaling functions for W_n depend sensitively on boundary conditions. Typical results of W_n and their finite-size scaling functions are reproduced in Fig. 10(a) and (b), respectively.

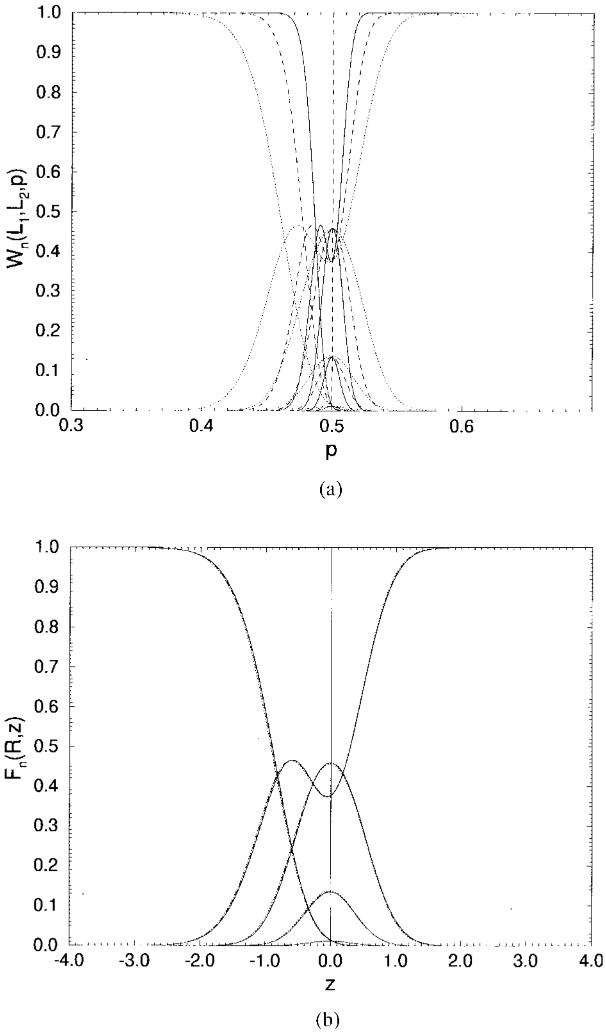


Fig. 10. (a) $W_n(L_1,L_2,p)$ for bond percolation on 128×32, 256×64 and 512×128 sq lattices, which are represented by dotted, dashed, and solid lines, respectively. (b) The data from (a) are plotted as a function of $z=(p-p_c)L^{1/\nu}$. The scaling function for $W_n(L_1,L_2,p)$ is denoted by $F_n(R,z)$, where $R=L_1/L_2$. The monotonic decreasing function is for $F_0(R,z)$. The S shape curve is for $F_1(R,z)$. The bell shape curves from the top to bottom are for $F_n(R,z)$ with n being 2, 3, and 4, respectively.

Using the HMCSM (Hu, 1992b, 1996b), Hu and Lin (1996) calculated W_n for bond and site percolation on sq, hc, and pt lattices with pbc in the horizontal direction and fbc in the vertical direction; the aspect ratios of sq, hc, and pt lattices are 4, $4\sqrt{3}$, $2\sqrt{3}$, respec-

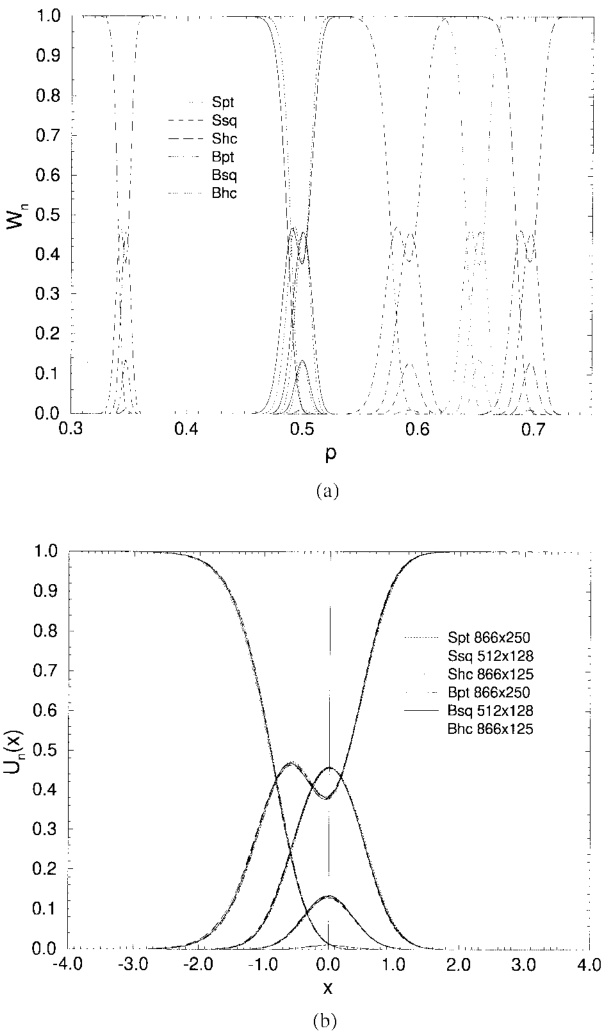


Fig. 11. (a) W_n for bond and site percolation on 866×250 pt, 512×128 sq, and 866×125 hc lattices. (b) The data in (a) are plotted as a function of $x=D_1(p-p_c)L^{1/\nu}$. The universal scaling function for W_n is denoted by $U_n(x)$.

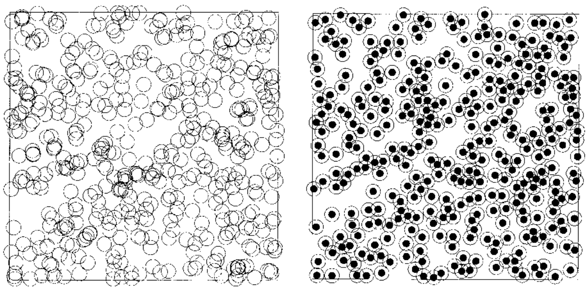
tively. Using nonuniversal metric factors of Hu *et al.* (1995a), Hu and Lin (1996) found that these percolation models have UFSSF's for W_n. The results for W_n and their UFSSF's are reproduced in Fig. 11(a) and (b), respectively.

Hu and Halperin (1997) considered bond percolation with bond probability p on an $L_1 \times L_2$ self-dual square lattice with pbc in the horizontal direction and fbc in the vertical direction. Hu and Halperin (1997) defined the number M of alternating percolating clusters as the minimum of n_{p} and n_{n}, where n_{p} is the number of independent percolating clusters connecting sites on the top and bottom edges, and n_{n} is the number of percolating clusters in the complementary configuration on the dual lattice, a bond being present in the complementary configuration if and only if it is absent in the original configuration. They used the HMCSM (Hu, 1992b, 1996b) to evaluate the probability $W_M^{\mathrm{a}}(L_1,L_2,p)$ of finding a given value of M and found that, for a given aspect ratio L_1/L_2, all data of $W_M^{\mathrm{a}}(L_1,L_2,p)$ near the critical point p_c fall on the same scaling function F_M^{a}, which is symmetric with respect to the scaling variable for all M. The values of $W_M^{\mathrm{a}}(L_1,L_2,p)$ at the critical point are useful for understanding $\sigma_{xx}^{\max}$ in the quantum Hall effects (Ruzin *et al.*, 1996; Hu and Halperin, 1997; Cooper *et al.*, 1997).

V. UFSSF's for Continuum Percolation of Disks

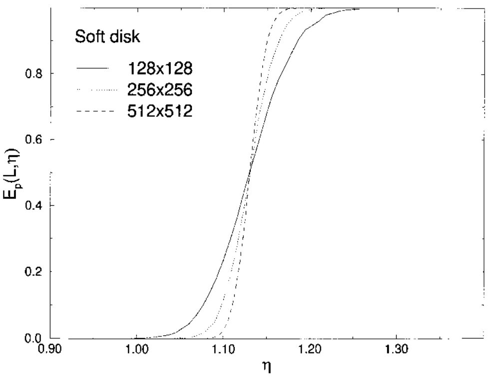
Many interesting quantities and problems in solid state physics, e.g., $\sigma_{xx}^{\max}$ in QHE, conductor-insulator transition, etc., are related to continuum percolation (Ruzin *et al.*, 1996; Zallen, 1983). However, to study continuum percolation is much more difficult than to study lattice percolation. People usually study lattice percolation rather than continuum percolation. The problem is to what extent the quantities, e.g., critical exponents and finite-size scaling functions, obtained from lattice percolation models (LPM) may be applied to continuum percolation models (CPM). Hu and Wang (1997) have tried to answer this interesting and important question.

Hu and Wang (1997) considered both soft disks and hard disks. Typical configurations of soft disks and hard disks are shown in Fig. 12(a) and (b), respectively. In the general case, Hu and Wang (1997) considered (hard and soft) disks on an $L_1 \times L_2$ continuum space C of rectangular domain with linear dimension L_1 in the horizontal direction and linear dimension L_2 in the vertical direction, where L_1 and L_2 are integers. The space C is divided into $L_1 \times L_2$ covering meshes, which are (1×1) unit squares. The squares (meshes) are labeled by integers 1, 2, 3, ..., $L_1 \times L_2$. A disk belongs to a square if and only if the center of the disk is in that square. The disks have a radius $R=\sqrt{2}/2$, so that at most one hard disk is allowed in one unit square. Two hard disks are in the same cluster if and only if their separation is smaller than or equal to $2\sqrt{2}$. Such a definition of clusters was considered by Hu (1987) and Kratky (1988) before. Two soft disks are in the same cluster if and only if they overlap. More than one soft disk may be in a given unit square, in which case they are always in the same cluster. Hu and Wang (1997) extended the multiple-labeling technique of Hoshen and Kopleman (1976) to label unit squares which have disks. The label for a unit square is also the label for the disks which belong to that unit square.

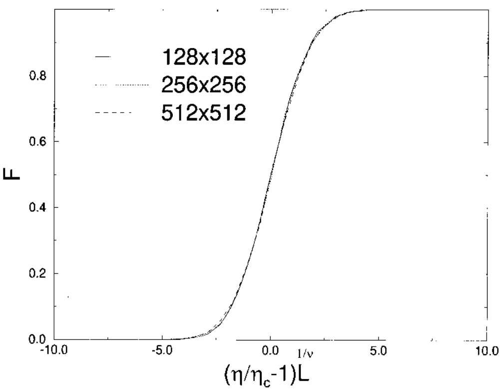


(a) (b)

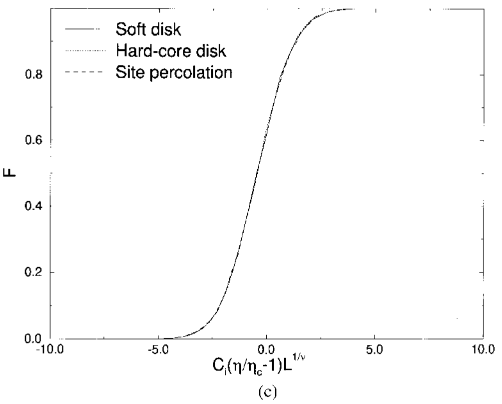
Fig. 12. (a) A configuration of soft disks for continuum percolation. (b) A configuration of hard disks for continuum percolation.



(a)



(b)



(c)

Fig. 13. (a) The calculated E_p as a function of η for continuum percolation of soft disks in an $L\times L$ space with free boundary conditions in the horizontal and vertical directions, where L=128, 256, and 512. The number of different η is between 50 and 100. The number of independent configurations for L=128, 256, and 512 is 40000, 10000, and 5000, respectively. (b) The data in (a) are plotted as a function of the scaling variable $x=(\eta-\eta_c)L^{1/\nu}$, where ν=4/3. The scaling function is $F(x)$. (c) The universal finite-size scaling function of E_p for soft disks, hard disks and site percolation on a square lattice. The number of independent configurations for hard disks and site percolation is two and eight times of that for soft disks, respectively. The non-universal metric factors for soft disks, hard disks, and lattice site percolation are C_1=1, C_2= 0.897±0.029, and C_3=1.60±0.07, respectively.

This multiple-labeling technique for CPM was used to study critical properties and scaling functions for soft disks and hard disks.

Hu and Wang (1997) used a random deposition process to generate configurations of disks. In the deposition process, if the hard cores of two disks overlap, then the attempt to put the second hard disk is abandoned. The concentration of disks is defined by $\eta=\pi R^2 N/L^2$, where N is the number of the disks in the system and $L=\sqrt{L_1 L_2}$ is the linear dimension of the system. At a given η, the number of percolating configurations observed divided by the total number of generated configurations gives the existence probability E_p. The calculated E_p as a function of η for continuum percolation of soft disks in $L\times L$ space with free boundary conditions in both the horizontal and vertical directions is reproduced in Fig. 13(a). The intersection of curves in Fig. 13(a) gives the critical point η_c and the critical existence probability $E_p(\eta_c)$, which are 1.1302±0.0008 and 0.50±0.01, respectively. The former is consistent with the result of Gawlinski and Stanley (1981), and the latter is consistent with the result of LPM (Ziff, 1992; Langlands *et al.*, 1992; Hu *et al.*, 1995a). From the slopes of curves at η_c. Hu and Wang (1997) used a percolation renormalization group method (Hu, 1992b) to find ν=1.39±0.07, which is consistent with the exact ν=4/3 for LPM on planar lattices (Stauffer and Aharony, 1994). The data in Fig. 13(a) as a function of the scaling variable $x=(\eta-\eta_c)L^{1/\nu}$ with ν=4/3 are reproduced in Fig. 13(b), which shows that E_p has very good scaling behavior. Hu and Wang (1997) obtained similar results for systems of hard disks and systems with pbc in the horizontal direction and fbc in the vertical direction. Typical results for these boundary conditions are reproduced in Fig. 13(c), which shows that E_p of soft disks, hard disks, and lattice site percolation have a universal finite-size scaling function.

Hu and Wang (1997) also calculated the probability W_n of the appearance of n percolating clusters for soft disks and hard disks in an $L_1 \times L_2$ space with pbc in the horizontal direction and fbc in the vertical direction. Typical calculated results are reproduced in Fig. 14(a)-(c). Figure 14(b) shows that W_n has very good scaling behavior, and Fig. 14(c) shows that W_n of soft disks, hard disks, and LPM have universal finite-size scaling functions. It is of interest to note that the nonuniversal metric factors in Fig. 14(c) are the same as those in Fig. 13 (c), which is similar to the case of lattice percolation (Hu and Lin, 1996).

To check the universality of critical exponents, Hu and Wang (1997) calculated the mean sizes of finite clusters $S(L,\eta_c)$, the mean sizes of percolating clusters $S_p(L,\eta_c)$, and the distributions of cluster sizes, $n(L,s,\eta_c)$, for soft disks, hard disks, and site percolation at their critical points η_c for systems of various linear dimension L. It follows from finite-size scaling and the scaling behavior of $n(L,s,\eta_c)$ (Stauffer and Aharony, 1994) that $S(L,\eta_c)\sim L^{\gamma/\nu}$, $S_p(L,\eta_c)\sim L^{d_f} = L^{d-\beta/\nu}$, $n(L,s,\eta_c)\sim s^{-\tau(L)}$. The critical exponents estimated from these equations are reproduced in Table 1, which shows that soft disks, hard disks, and percolation on planar lattices are in the same universality class.

VI. UFSSF's for a Correlated Percolation Model

Based on the subgraph expansion of Ising-type models in external fields, Hu has shown that phase transitions of many Ising-type models can be described as geometric percolation transitions (Hu, 1984a, 1984b, 1988, 1990, 1992a). In particular, Hu has shown that phase transitions of QPM on a lattice G are percolation transitions of QBCPM (Hu, 1984a, 1984b, 1988, 1990, 1992a) on G, in which each NN bond of G is occupied by a probability p, where $p=1-\exp(-J/k_BT)$ with J being the ferromagnetic Potts coupling constant. Sites connected by occupied bonds are in the same cluster, and a cluster may have any one of q different directions. There are 2^E different bond configurations G', also called "subgraphs", of G. A subgraph G' of $b(G')$ occupied bonds and $n(G')$ clusters will appear with the probability weight

$$\pi(G',p,q)=p^{b(G')}(1-p)^{E-b(G')}q^{n(G')}.$$

The spontaneous magnetization and the magnetic susceptibility of the QPM are related to the percolation probability P and the mean cluster size of the QBCPM, respectively (Hu, 1984a, 1984b). The probability of the appearance of percolating clusters, E_p, of the QBCPM is defined by

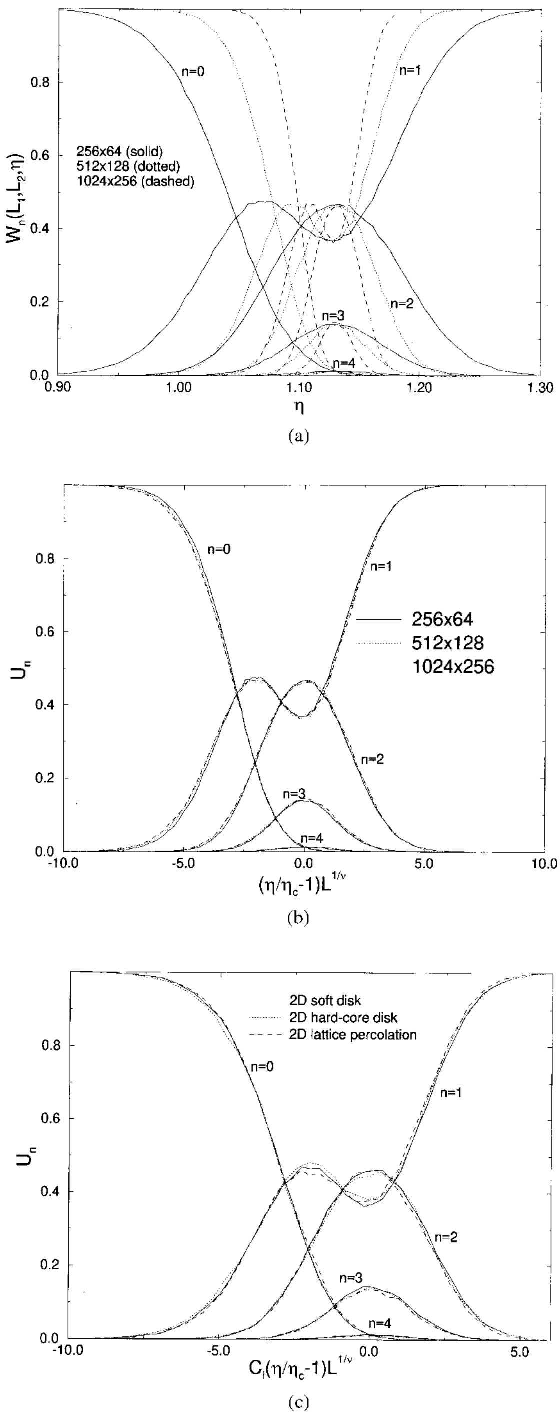


Fig. 14. (a) $W_n(L_1,L_2,\eta)$ as a function of η for continuum percolation of soft disks in 256×64, 512×128 and 1024×256 space, which is represented by solid, dotted, and dashed lines, respectively. (b) The data in (a), $W_n(L_1,L_2,\eta)$, as a function of $z=(\eta-\eta_c)L^{1/\nu}$. The scaling function for $W_n(L_1,L_2,\eta)$ is denoted by U_n. (c) The universal finite-size scaling functions of W_n with n=0, 1, 2, 3, 4 for soft disks, hard disks, and site percolation on square lattice systems. The non-universal metric factors for soft disks, hard disks, and lattice site percolation are C_1=1, C_2=0.897±0.029, and C_3=1.60±0.07, respectively.

Table 1. Universality of $E_p(\eta_c)$ and Critical Exponents for 2D Continuum Percolation. For E_p, We Consider Both Free Boundary Conditions (fbc) and Periodic Boundary Conditions (pbc) in the Horizontal Direction

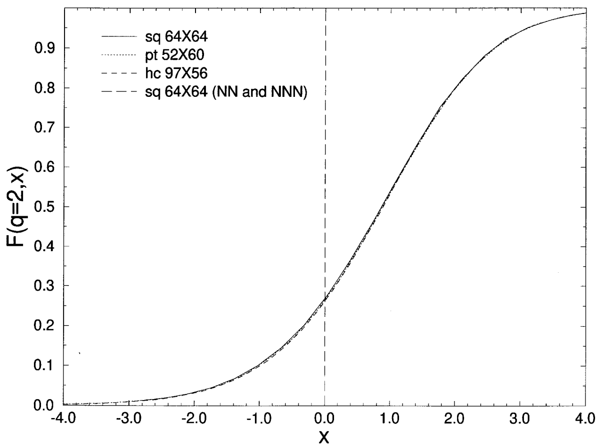
quantities	soft disks	hard disks	LPM	exact
threshold	1.1302±0.0008	0.8503±0.0010	0.5928±0.0010	
$E_p(\eta_c)$ (fbc)	0.50±0.01	0.50±0.03	0.50±0.01	0.5
$E_p(\eta_c)$ (pbc)	0.64±0.02	0.64±0.02	0.63±0.02	
ν	1.39±0.07	1.36±0.04	1.37±0.05	1.33...
γ/ν	1.785±0.012	1.790±0.012	1.780±0.020	1.7916...
$d-\beta/\nu$	1.889±0.006	1.885±0.008	1.891±0.016	1.89583...
τ	2.05±0.02	2.05±0.02	2.04±0.02	2.0549...

$$E_p(G,p,q)=\frac{\sum_{G'_p\subseteq G}\pi(G'_p,p,q)}{\sum_{G'\subseteq G}\pi(G',p,q)}. \quad \textbf{(16)}$$

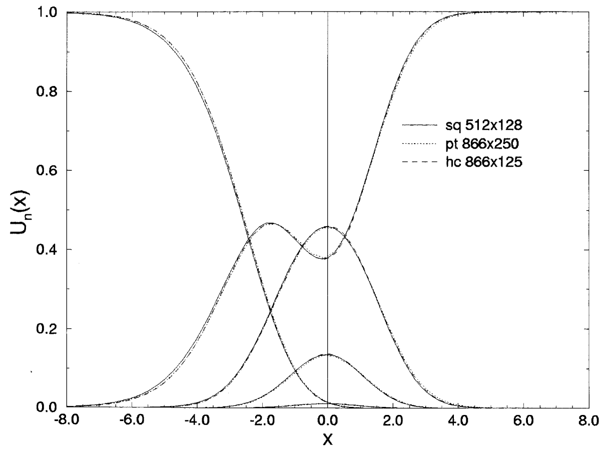
Here the sum in the denominator is over all subgraphs G' of G, and the sum in the numerator is restricted to all percolating subgraphs G'_p of G.

Hu *et al.* (1999a) used a cluster Monte Carlo simulation method (Swendsen and Wang, 1987; Hu and Mak, 1989b) to simulate the QBCPM and QPM on a 64×64 sq lattices, a 97×56 hc lattices, and a 52×60 pt lattices. The aspect ratios of these lattices approximately match the relative proportions of Langlands *et al.* (1992). Typical calculated results of E_p as a function of the scaling variable x with $x=tL^{1/\nu}$ $(t=(T-T_c)/T)$ is presented in Fig. 15(a), which shows that E_p for the QBCPM and QPM on sq, hc, and pt lattices have UFSSF near x=0 *without* nonuniversal metric factors. For the sq lattice, we also consider a model with both NN and NNN coupling and find that the calculated E_p as a function of x have the same FSSF as does the model without NNN coupling. Since the QPM for q=1 corresponds to the bond random percolation model (BRPM), we also plot data of W_n from Hu and Lin (1996) for bond percolation on sq, hc, and pt lattices as a function of $x=tL^{1/\nu}(t=(T-T_c)/T)$ and obtain Fig. 15(b), which shows that we have UFSSF's for W_n without using nonuniversal metric factors. These results are analogous to setting C_1=1 in PF's theory (see Eq. (2) in this paper) for all lattices of a given dimension. We also find similar results for three dimensional lattices.

It should be noted that if we use the scaling variable $z=(p-p_c)L^{1/\nu}$ or $z'=(p/p_c-1)L^{1/\nu}$ as the horizontal axis, then we need to use metric factor for each lattice. This suggests that $t=1-T/T_c$ is a fundamental variable for describing critical phenomena near the critical point, even for BRPM (Hu *et al.*, 1999a).



(a)



(b)

Fig. 15. (a) E_p for the QBCPM as a function of $x=(1-T/T_c)L^{1/\nu}$. The scaling function is $F(q,x)$. For the sq lattice, the results for a model with both NN and NNN couplings is also shown. Here q=2. (b) W_n for the QBCPM as a function of $x=(1-T/T_c)L^{1/\nu}$. The scaling function for W_n is U_n. Here q=1.

VII. Boundary Conditions and the Number of Percolating Clusters

The universality of W_n implies that the average number of percolating clusters, defined by $C=\sum_{n=1}^{\infty} nW_n$, is also universal. In Hu (1996b) and Hu and Lin

(1996), the critical C was calculated for percolation on lattices with BC1 and BC2 for aspect ratio R between 0 and 10, where $R=L_1/L_2$. They found that for large R, the critical C increases linearly with R with the same slope.

In a recent letter, Ziff *et al*. (1997) calculated the number of clusters per lattice site, n, in bond and site percolation on two dimensional lattices with BC3 and the linear dimension L. They found that $n=n_c+b/N$, where n_c is n in the limit $L\to\infty$, b is a constant and N is the number of lattice sites. They also found that b is universal and presented an argument that b is the number of percolating clusters so that the universality of b may be related to the universality of C found by Hu and Lin (1996). Kleban and Ziff (1998) obtained an exact formula for b as a function of the aspect ratio, R, which agrees very well with the numerical result.

In a recent paper, Hu (1999) used HMCSM to calculate n and C for bond percolation on $L_1\times L_2$ sq lattices with BC1, BC2, BC3, and BC4. In BC3 and BC4, a cluster is percolating if each of L_2 rows contains at least one site of that cluster (Hu, 1996a). Hu found that for four different boundary conditions, C increases linearly with R with approximately the same slope. On the other hand, we may have a well defined slope b in $n=n_c+b/N$ only for BC3.

Now we will briefly review HMCSM for the calculation of the number of clusters per site, $n(L_1,L_2,p)$, for bond percolation on an $L_1\times L_2$ sq lattice. Let $M_c(G')$ denote the total number of clusters in subgraph G'. Then $n(L_1,L_2,p)$ for bond percolation with a bond probability p is given by

$$n(L_1, L_2, p) = \sum_{G' \subseteq G} p^{b(G')}(1-p)^{E-b(G')}M_c(G')/N \,. \tag{17}$$

The sum in Eq. (17) is over all subgraphs G' of G. To use HMCSM to evaluate $n(L_1,L_2,p)$, in addition to $N_p(b)$ and $N_f(b)$ considered in Sec. II, we also evaluate $N_c(b)$, $0\le b\le E$, which is total number of clusters in the subgraphs with b occupied bonds. After a large number of simulations $n(L_1,L_2,p)$ at any value of the bond occupation probability p can be calculated approximately from the following equation:

$$n(L_1, L_2, p) = \frac{1}{N}\sum_{b=0}^{E} p^b(1-p)^{E-b}C^E_b\frac{N_c(b)}{N_p(b)+N_f(b)} \,. \tag{18}$$

The calculated critical points $n(L_1,L_2,p)$ as a function of $1/N$ for bond percolation on $L\times L$ lattices with BC3 are shown in Fig. 16(a), which is consistent with the result of Ziff *et al*. (1997). The calculated b and C for bond percolation on $16R\times16$ lattices with BC3 are plotted in Fig. 16(b), which shows that the curves for b and C for large R linearly increase with R, but that they have different slopes (Hu, 1998). The calculated C for bond percolation on $16R\times16$ lattices with BC1 to BC4 are plotted in Fig. 16(c), which shows that for large R, C linearly increases with R, and that all the curves have the same slope.

VIII. Some Related Developments

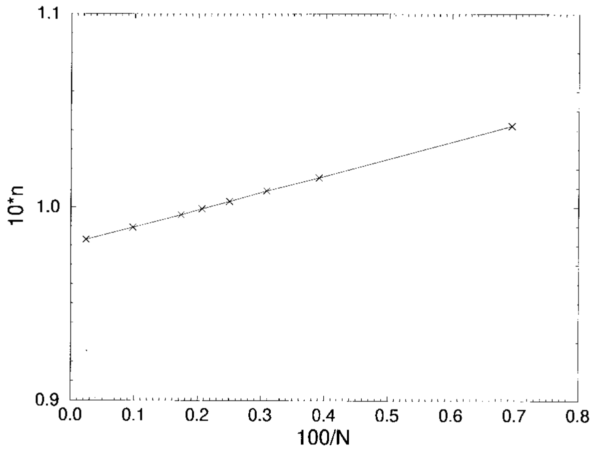
Recently, Okabe *et al*.[1] calculated FSSF's of the Binder parameter g and magnetization distribution function $p(m)$ for the Ising model on $L_1\times L_2$ square lattices with pbc in the horizontal L_1 direction and tilted boundary conditions with tilt parameter c in the vertical L_2 direction. For appropriate sets of (R,c) with $R=L_1/L_2$, the FSSFs of g and $p(m)$ are universal, and in such cases $R/(c^2R^2+1)$ is invariant. For percolation on lattices with fixed R, FSSF of the existence probability does not change as c increases from 0.

Very recently, Hu *et al*. (1999b) used a cluster Monte Carlo method to calculate the number of clusters per site, n, at the critical point of the QBCPM on $L'\times L$ sq lattices. Typical results for q=1 are shown in Fig. 17(a), and those for q=2, 3, and 4 are shown in Fig. 17(b). Figure 17(a) is consistent with the result of Ziff *et al*. (1997). However, curves in Fig. 17(b) have negative slopes, which are quite different from those in Fig. 17(a), and the interpretation of slope as the number of percolation clusters (Ziff *et al*., 1997) is impossible. To understand the behavior of Fig. 17(b), Hu *et al*. (1999b) proposed that n as a function of $1/L$ for $q\neq2$ and fixed L'/L has an energy-like singularity. For q=2, i.e. the Ising model, they found that the data can be well represented by $n=n_c-c/L+b/L^2+\ldots$, where b can be calculated exactly from conformal field theory (CFT), $c>0$ and can be calculated exactly from a formula for the internal energy of the Ising model. A typical comparison of the formula and numerical data is shown in Fig. 18. The agreement between the formula and the data is very good.

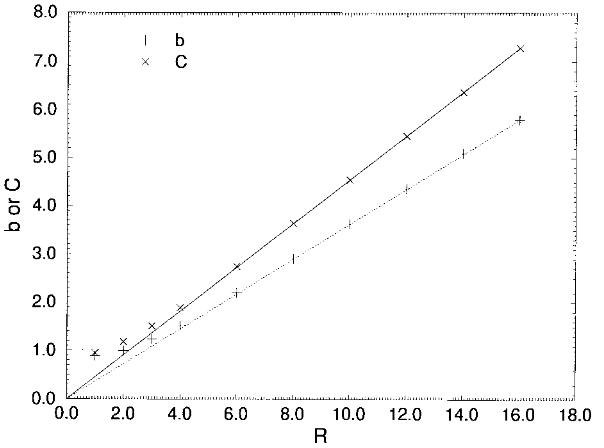
IX. Summary and Final Remarks

Using HMCSM (Hu, 1992b, 1996b) and relative aspect ratios considered by Langlands *et al*. (1992), we found universal finite-size scaling functions for the

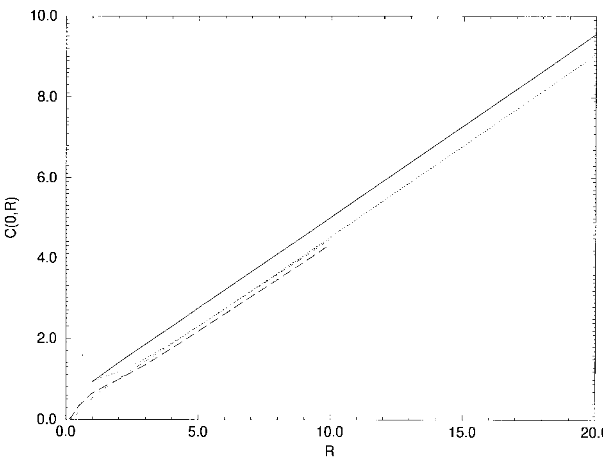
[1]Okabe, Y., K. Kaneda, M. Kikuchi, and C. K. Hu, "Universal finite-size scaling functions for critical systems with tilt boundary conditions." Sunmitted to *Phys. Rev. Lett.*



(a)

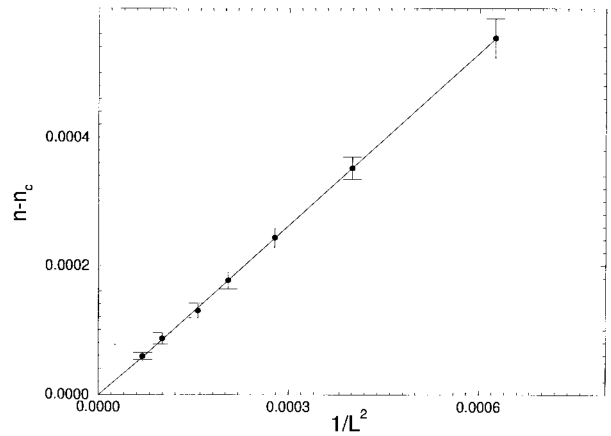


(b)

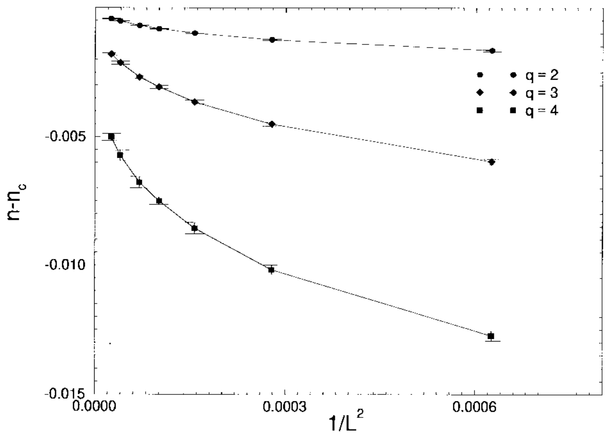


(c)

Fig. 16. (a) The number of clusters per site, n, as a function of $1/N$ for bond percolation on $L{\times}L$ lattices with BC3, where $N{=}L^2$. (b) b and C as a function of R for lattices with BC3. (c) $C(0,R)$ as a function of R for lattices with different boundary conditions. Near R=3, the curves from bottom to top are for BC1, BC2, BC3, and BC4, respectively.



(a)



(b)

Fig. 17. $n{-}n_c$ as a function of $1/L^2$ for the q-state bond correlated percolation model on $L{\times}L$ square lattices with periodic boundary conditions (torus) for (a) q=1, (b) q=2, 3, and 4. n_c is n in the limit $L{\to}\infty$.

existence probability, E_p, the percolation probability, P, and the probability for the appearance of n percolation clusters, W_n, of site and bond percolation on sq, hc, and pt lattices. Using a random deposition process, we found UFSSF's for E_p and W_n for the CPM of soft disks and hard disks and LPM. Table 1 shows that the CPM of soft disks and hard disks are in the same universality class as the lattice percolation models.

We may consider the general case in which a disk has a hard core of radius R_1 and a soft shell of radius R_2, where $R_1{<}R_2$. The soft disk in Fig. 12(a) corresponds to R_1=0, and the hard disk in Fig. 12(b) corresponds to $R_1{=}R_2/2$. Two disks are in the same cluster if their soft shells overlap. The general case $0{\le}h{=}R_1/R_2{<}1$ was considered by Lee (1990). However, he did not reach a definite result about the universality of such general hard disks. Our results show that disks

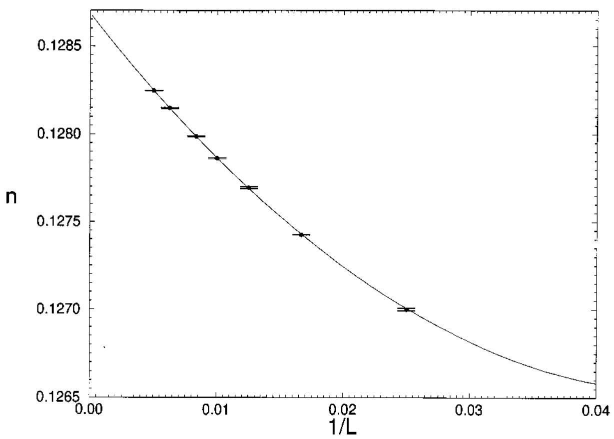


Fig. 18. n as a function of $1/L$ for the Ising model on $L\times L$ square lattices with periodic boundary conditions (torus). The solid line represents $n=n_c-c/L+b/L^2+...$, where b can be calculated exactly from conformal field theory, $c>0$ and can be calculated exactly from exact formula for the internal energy of the Ising model.

with $h=0$ and $h=0.5$ are in the same universality class, which suggests that disks with $0\leq h<1$ may be in the same universality class. Further studies in this direction are needed.

In Sec. V, we found that continuum percolation of soft disks and hard disks has the same UFSSF's as does percolation on planar lattices. We expect that the data of C for continuum percolation should collapse with the data of lattice percolation shown in Fig. 16(c) when both models have the same aspect ratio and boundary conditions.

For BC3, if we consider only percolating clusters which contain closed loops, then the value of C might be smaller than the value shown in Fig. 16(b), and the linear slope of C might be closer to the linear slope of n.

For random percolation, Cardy (1992, 1998) used CFT to obtain exact formula for E_p for any R and W_n for large R; the former agrees very well results of Langlands *et al.* (1992). It would be valuable to extend such analytic and numerical calculations of E_p and W_n to the QBCPM. Figure 15 represents an effort in this direction. Figure 15 suggests that $t=1-T/T_c$ is more fundamental than $p-p_c$ or p/p_c-1. A physical interpretation of T in the BRPM is needed.

Our results suggest that computer simulations can help us to understand the mysteries of nature, and that the physics of critical phenomena in finite systems is interesting and simple.

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展透模型之普適尺度函數

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摘　要

許多數學、物理及生命科學的問題可用展透模型描述。本論文回顧筆者及合作者近年來在展透模型普適量及普適尺度函數所作成的研究。我們研究的量包括存在或然率E_p、展透或然率P、出現n個展透集團的機率W_n等。本論文評述的主題包括：

⑴系統的邊界條件和寬高比與尺度函數的關係
⑵晶格展透模型E_p和P的普適尺度函數
⑶晶格展透模型W_n的普適尺度函數
⑷連續展透模型E_p和W_n的普適尺度函數
⑸q狀態相關展透模型之普適尺度函數
⑹邊界條件與平均展透數

本論文最後提及一些相關的發展及值得進一步研究的問題。