Proc. Natl. Sci. Counc. ROC(A)
Vol. 21, No. 6, 1997. pp. 591-600

A Pallet Loading Method for Single-size Boxes with Optimal Stability

FUH-HWA LIU AND CHIH-JEN HSIAO

Department of Industrial Engineering and Management
National Chiao Tung University
Hsinchu, Taiwan, R.O.C.

(Received January 30, 1997; Accepted May 20, 1997)

ABSTRACT

The problem of finding an optimal loading layout for packing identical boxes onto a pallet is known as the pallet loading problem. If the boxes are stacked on their bottom, side, or end surfaces, the cube utilization of a pallet will increase, but the stability of the unit load may drop. To quantify the stability of the unit load, this paper defines the stability coefficient between any two adjoined layers. According to the stability coefficient, a method with five phases that packs boxes of the same size onto one pallet is proposed. The objective of this method is to maximize the smallest stability coefficient of the interface in the unit load based on the maximum cube utilization of a unit load. An example illustrates how the method works, and the method has been tested using 216 box sizes and 3 standard pallets combinations.

Key Words: distribution, logistics, packing, pallet loading

I. Introduction

The problem of finding an optimal loading layout for packing identical boxes onto a pallet - the so-called pallet loading problem - arises frequently in logistics. In the pallet loading problem, the size $l \times w$ of the box and the size $L \times W$ of the pallet are the two major parameters used to determine the maximum number of boxes that can be loaded on the pallet. These methods may produce solutions that do not satisfy real-life problems such as load stability. Therefore, Carpenter and Dowsland (1985) proposed the *supportive criterion*, *base contact criterion*, and *non-guillotine criterion* to ensure the stability of pallet loading patterns that maximize the number of boxes loaded.

Bischoff (1991) considered the stability objective in the pallet loading problem and examined approaches for generating stable stacking patterns that are also optimal for area utilization. By using *compacting*, *centering blocks* and *distributing gaps* procedures, Bischoff extended the Bischoff-Dowsland algorithm (Bischoff and Dowsland, 1982) to generate more stable layouts under the same criteria proposed by Carpenter and Dowsland (1985).

Liu and Hsiao (1997) integrated the methods for cube utilization and the criteria for stability, and assumed that the stability of the unit load is the sum of the stability coefficients of every interface from the highest layer to the lowest layer. A five-phase method was proposed to determine the loading patterns and the stacking sequence which had the highest level of stability while achieving the maximum number of boxes on the pallet. This method provides the best cube utilization and stability.

From a practical point of view, the smallest value of the stability coefficients in the unit load is the weakest interface that influences the stability of the unit load. Extending our previous work, a max-min model, where a solution is sought such that the smallest value of the stability between two adjoining layers is as high as possible, is proposed in this paper.

Since the max-min model is more complex than the max-sum model, mathematical programming computer software packages can not be used to solve the IP Code directly. In this paper, we propose two upper bounds and one lower bound for the max-min model. Using these bounds and the characteristic of the stability coefficient, the max-min model can be solved in reasonable time.

In this paper, we present a new method for computing the stability coefficient between any two adjoining layers. This method considers characteristics such as the relative position of the box on the pallet, the number of contacts in the Supportive criterion, and the size of the contact area in the Base contact criterion.

II. Definition of Stability Coefficients

Different methods may increase the stability of a unit load with different costs. Using an analytical approach to generate optimum loading patterns results in lower costs than do other methods such as stretch wrapping, shrink wrapping, strapping, gluing, and applied tie sheets (Cox and Van Tassel, 1985). To find the best loading pattern, we quantify the stability criteria and define the stability coefficients between adjoining layers.

Usually, a pallet is loaded with multiple layers. The interfaces of the adjoining layers affect the stability of the whole unit load. The principles for defining the stability coefficients of the adjoining layers are as follows:

(1) One layer stacked on another layer has a certain stability.
(2) For two adjoining layers, the stability of the upper layer depends on its loading pattern and that of the lower layer as well.
(3) The stability of a layer is determined by the stability of every box in the layer.
(4) Because boxes on the corner and perimeter of a pallet are in contact with the forklift, it is prudent to pay more attention to these areas to reduce any possible damage.
(5) A box's center of gravity is on its center of geometry.
(6) The stability of two adjoining layers can be evaluated by means of the stability criteria.

Let ω_{ij} represent the stability coefficient of the interface when an i-pattern layer is stacked on a j-pattern layer. The definition of the stability coefficient, ω_{ij}, is based on the Supportive criterion and Base contact criterion (Carpenter and Dowsland, 1985).

(1) **Supportive criterion:** If the base of a box k in an i-pattern layer is in contact with at least two boxes in the j-pattern layer, and each contact must include more than $\sigma_s\%$ of a box's base area, we say that this box satisfies the Supportive criterion.
(2) **Base contact criterion:** If at least $\sigma_b\%$ of its base area of a box k in an i-pattern layer has is in contact with the layer below, we say that this box satisfies the Base contact criterion.

The Supportive criterion tends to tie boxes together by bridging boxes. Let CN_k be the number of boxes in the lower layer that are in contact with box k in the upper layer, for example, (see Fig. 1), CN_{k1}=4, CN_{k2}=2. The arrangement with the higher CN_k value is preferred. Depending on the box's dimensions, we can specify a threshold value, CN^*, for the Supportive criterion.

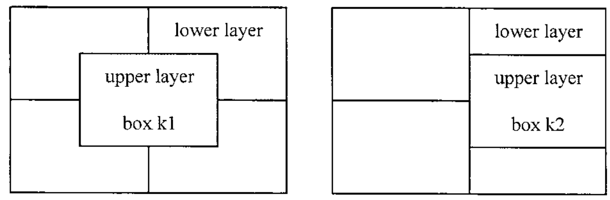

Fig. 1. The stacking status for the Supportive criterion.

The Base contact criterion tends to increase the area of the box in contact with the layer below. Let CA_k be the area of box k in the upper layer in contact with boxes in the lower layer. The higher CA_k is preferred.

Since the locations of the boxes in a layer play a significant role in their stability, we assume that their stability is proportionally weighted by their distance to the center of the pallet. Let d_k be the weighting factor of box k for computing the stability coefficient. d_k is computed as the distance between the center of box k and the center of the pallet. Hence, the corner box will have more stability weighting.

As mentioned above, the stability coefficient, ω_{ij}, can be defined as follows:

$$\omega_{ij} = \begin{cases} \dfrac{\sum_{k=1}^{Q_i} d_k\left(\rho_s \dfrac{CN_k}{CN^*} + \rho_b \dfrac{CA_k}{CA^*}\right)}{\sum_{k=1}^{Q_i} d_k} & \forall \dfrac{CA_k}{CA^*} \geq \sigma_b\% \\ -\infty & \exists \dfrac{CA_k}{CA^*} < \sigma_b\%, \end{cases} \tag{1}$$

where

Q_i is the total number of boxes in the i-pattern layer,

ρ_s is the weighting factor of the Supportive criterion,

ρ_b is the weighting factor of the Base contact criterion,

$\rho_s+\rho_b=1$,

CA^*=the area of the box.

III. A Method for Pallet Loading

Boxes with dimensions $l\times w\times h$ have to be loaded onto a pallet of size $L\times W$, subject to a height limit of H. The unit load will be handled by a forklift with weight capacity M. The objective of the method is to maximize the degree of stability while achieving the maximum cube utilization of a unit load under the following assumptions:

(1) the boxes may be stacked on their bottom, side, or end surfaces;
(2) all the boxes within a layer must be stacked on the same surface;
(3) the degree of stability between any two adjoining layers is determined by the stability coefficient.

1. Phase 1: Determining the Maximum Number of Boxes for Each of the Three Possible Types of Layers.

Three loading types are defined as follows:

(1) B-type layer: a layer of boxes with a height of h (and bottom surface $l\times w$);
(2) S-type layer: a layer of boxes with a height of w (and side surface $l\times h$);
(3) E-type layer: a layer of boxes with a height of l (and end surface $w\times h$).

There are many approaches (Steudel, 1979; Smith and de Cani, 1980; Bischoff and Dowsland, 1982) that may achieve better area utilization for the three basic loading types. Any effective approach can be applied to generate B-, S-, and E-type layers in this phase.

Let the maximum number of boxes in a B-type layer be denoted by N_B, the maximum in an S-type layer by N_S and the maximum in an E-type layer by N_E. Let the associated loading patterns for B-, S-, and E-type layers be Bo, So, and Eo, respectively.

2. Phase 2: Constructing Three Related Patterns for Each Type of Layer.

Each of the three basic loading patterns obtained in Phase 1 is transformed by means of the following methods (Carpenter and Dowsland, 1985):

(1) α-transformation: mirror reflection along one side of the pattern;
(2) β-transformation: mirror reflection along the other side of the pattern;
(3) γ-transformation: 180 degree rotation on the same plane of the pattern.

Hence, the Bo pattern is transformed into patterns $B\alpha$, $B\beta$, and $B\gamma$. Similarly, So is transformed into $S\alpha$, $S\beta$, and $S\gamma$, and Eo is transformed into $E\alpha$, $E\beta$, and $E\gamma$, for a total of twelve patterns that may be selected for stacking.

3. Phase 3: Finding the Maximum Number of Boxes That Can Be Included in the Loading Cube of the Pallet.

Let variables Z_B, Z_S, and Z_E represent the number of B-, S-, and E-type layers stacked on the pallet, respectively. The maximum number of boxes (Λ^*) loaded on the pallet can be found by solving the following integer programming (IP) problem:

$$\Lambda^*=\text{Max. } (N_B\times Z_B+N_S\times Z_S+N_E\times Z_E), \tag{2}$$

subject to

$$h\times Z_B+w\times Z_S+l\times Z_E\leq H \tag{3}$$

$$N_B\times Z_B+N_S\times Z_S+N_E\times Z_E$$

$$\leq \text{Min}[\lfloor M/m\rfloor, \lfloor (L\times W\times H)/(l\times w\times h)\rfloor], \tag{4}$$

where m is the weight of the box and Z_B, Z_S, Z_E are integers.

Equation (3) ensures that the stacking height does not exceed the limit of the unit load. The weight and volumetric limitations are presented in Eq. (4).

4. Phase 4: Computing the Stability Coefficients of 144 Possible Interfaces.

Using Eq. (1), 144 stability coefficients are computed using all possible combinations of twelve possible layer patterns provided by Phase 2. Let $\mathbf{\Omega}$ be the matrix that comprises the stability coefficients (ω_{ij}) of all possible interfaces.

5. Phase 5: Determining the Stacking Sequence of the Pattern That Will Construct a Unit Load with Maximum Stability.

To obtain the maximum stability of a unit load, the number of layers of each possible pattern for each type and the stacking sequence of the layers must be determined in this phase.

Every possible stacking sequence has $(Z_B+Z_S+Z_E)$-1 interfaces between layers. Let y_j be the number of times an i-pattern layer is included in the unit load, and x_{ij} be the number of interfaces that the i-pattern layer stacks on the j-pattern layer. The constraints for determining the stacking sequence are as follows:

$$y_{Bo}+y_{B\alpha}+y_{B\beta}+y_{B\gamma}=Z_B, \tag{5}$$

$$y_{So}+y_{S\alpha}+y_{S\beta}+y_{S\gamma}=Z_S, \tag{6}$$

$$y_{Eo}+y_{E\alpha}+y_{E\beta}+y_{E\gamma}=Z_E, \tag{7}$$

$$\sum_{j\in\mathbf{P}} x_{ij} + x_{iV} = y_i \ (\forall i\in\mathbf{P}), \tag{8}$$

$$\sum_{i\in\mathbf{P}} x_{ij} + x_{Vj} = y_j \ (\forall j\in\mathbf{P}), \tag{9}$$

$$\sum_{j\in\mathbf{P}} x_{Vj} = 1, \tag{10}$$

$$\sum_{i\in\mathbf{P}} x_{iV} = 1, \tag{11}$$

$$\mathbf{X}=(x_{ij})\in\mathbf{G}, \ (\forall i,j\in\mathbf{P}), \tag{12}$$

where

$\mathbf{B}=\{Bo, B\alpha, B\beta, B\gamma\}$, $\mathbf{S}=\{So, S\alpha, S\beta, S\gamma\}$,

$\mathbf{E}=\{Eo, E\alpha, E\beta, E\gamma\}$,

$\mathbf{P}=\mathbf{B}\cup\mathbf{S}\cup\mathbf{E}$,

$\mathbf{G}=\{x_{ij} | \sum_{i\in\mathbf{R}}\sum_{j\in\mathbf{R}} x_{ij} \le \sum_{i\in\mathbf{R}} y_i - 1$ for every nonempty subset $\mathbf{R}$ of $\mathbf{P}$ and $y_i>0\}$,

V represents the virtual layer of a unit load at the top or the bottom,

y_i is an integer, $\forall i\in\mathbf{P}$,

x_{ij} is an integer, $\forall i,j\in\mathbf{P}$,

x_{iV} is binary, $\forall i\in\mathbf{P}$,

x_{Vj} is binary, $\forall j\in\mathbf{P}$.

Equation (5) shows that the sum of the number of Bo-, $B\alpha$-, $B\beta$-, and $B\gamma$-patterns must equal Z_B. Equations (6) and (7) are the same constraints for S-type and E-type layers. Equation (8) shows that instances of the i-pattern have y_i interfaces with the layers beneath them, including the virtual layer. Equation (9) shows that instances of the j-pattern have y_j interfaces with the layers above them, including the virtual layer. Equation (10) indicates that there is only one layer beneath the virtual layer in the unit load. Similarly, Eq. (11) indicates that there is only one layer above the virtual layer in the unit load.

Equation (12) can eliminate the possibility of unsuitable outcomes and guarantee that the stacking sequence of the unit load will be a single non-simple path from V to V. The general expression of Eq. (12) can be written as Eqs. (13)-(15):

$$\sum_{i\in\mathbf{R}}\sum_{j\in\mathbf{R}} x_{ij} \le \sum_{i\in\mathbf{R}} y_i - 1 + \sum_{i\in\mathbf{R}} K(1-\lambda_i) \ (\forall\mathbf{R}\subset\mathbf{P}), \tag{13}$$

$$y_i \le K\lambda_i \ (\forall i\in\mathbf{R}), \tag{14}$$

$$y_i \ge \lambda_i \ (\forall i\in\mathbf{R}), \tag{15}$$

where λ_i is the auxiliary binary variable for $\forall i\in\mathbf{R}$ and K is a sufficiently large number.

Consider a unit load including t layers, where $t=Z_B+Z_S+Z_E$. Let $\omega_{ij}(r)$ indicate the stability coefficient of the interface such that the r-th layer in a stack is an i-pattern instance and the $(r+1)$-th layer is a j-pattern instance. The loading patterns of the unit load from the highest layer to the lowest layer are $p1, p2, ..., pt$. Hence, the $t-1$ stability coefficients in the sequence will be $\omega_{p1p2}(1)$, $\omega_{p2p3}(2)$, ..., $\omega_{p(t-2)p(t-1)}(t-2)$, $\omega_{p(t-1)pt}(t-1)$.

If we assume that the stability of the unit load, θ, is the sum of the stability coefficients of every interface from the highest layer to the lowest layer, then

$$\theta = \sum_{r=1}^{t-1} \omega_{ij}(r), \text{ where } t=Z_B+Z_S+Z_E. \tag{16}$$

The greatest stability of the unit load can be indicated by the stacking sequence with the maximum θ value:

Model O1: objective function

$$\theta^* = \sum_{r=1}^{t-1} \omega_{ij}^*(r) = \text{Max} \left(\sum_{i\in\mathbf{P}}\sum_{j\in\mathbf{P}} \omega_{ij} x_{ij}\right). \tag{17}$$

We also define that ω_{ave}^* is $\theta^*/(t-1)$. The max-sum model has been proposed by Liu and Hsiao (1997).

From a practical point of view, we want the smallest value of the stability coefficient between adjoining layers, π, to be as high as possible. Hence, we consider a max-min objective function with the same constraints as in Eqs. (5)-(12):

$$\pi=\text{Min}\{\omega_{p1p2}(1), \omega_{p2p3}(2), ..., \omega_{p(t-2)p(t-1)}(t-2), \omega_{p(t-1)pt}(t-1)\}. \tag{18}$$

Model O2 seeking the stacking sequence with maximum π value is as follows:

$$\text{Model O2: objective function } \pi^*=\text{Max} \min_{\omega_{ij}\in\Phi}(\omega_{ij}), \tag{19}$$

where Φ is the set of all feasible stacking sequences.

IV. The Solution Procedure for the Max-min Stability Model

To solve Model O2, we remodel it as Model O3, which is a general mixed-integer programming model. The objective function and the four additional constraints are:

Model O3: objective function Max. (π), **(20)**

subject to

$$\pi \leq \omega_{ij} u_{ij} + K(1-u_{ij}) \ (\forall i,j \in \mathbf{P}), \quad \textbf{(21)}$$

$$x_{ij} \leq K u_{ij} \ (\forall i,j \in \mathbf{P}), \quad \textbf{(22)}$$

$$x_{ij} \geq u_{ij} \ (\forall i,j \in \mathbf{P}), \quad \textbf{(23)}$$

$$\sum_{i \in \mathbf{P}} \sum_{j \in \mathbf{P}} u_{ij} \geq 1, \quad \textbf{(24)}$$

where u_{ij} is binary ($\forall i,j \in \mathbf{P}$) and K is a sufficiently large number.

Hence, 433 (144×3+1) additional constraints and 144 additional binary variables (u_{ij}) are incorporated. Because u_{ij} is the yes-no type variable for a decision, the model will be difficult to solve in a reasonable amount of time. We introduce three ways to reduce the complexity of the model in the following.

1. Reduce the Problem Size by Means of the Characteristic of the Stability Coefficient Matrix.

The stability coefficients matrix Ω can be decomposed into nine submatrices, $\Omega_{\mathbf{I}\times\mathbf{J}}$, where **I** and **J** are **B**, **S**, and **E**. The size of each submatrix is 4×4, with 16 entries.

Since the relative positions for some adjacent stacking layers are identical, their stability coefficients can be represented by one of them. For instance, in the submatrix $\Omega_{\mathbf{B}\times\mathbf{S}}$, $\omega_{BoSo}=\omega_{B\alpha S\alpha}=\omega_{B\beta S\beta}=\omega_{B\gamma S\gamma}$, Eqs. (21)-(23) for $(i, j)=\{(Bo, So), (B\alpha, S\alpha), (B\beta, S\beta), (B\gamma, S\gamma)\}$ can be simplified as follows:

$$\pi \leq \omega_{BoSo} u_{BoSo} + K(1-u_{BoSo}), \quad \textbf{(25)}$$

$$x_{BoSo}+x_{B\alpha S\alpha}+x_{B\beta S\beta}+x_{B\gamma S\gamma} \leq K u_{BoSo}, \quad \textbf{(26)}$$

$$x_{BoSo}+x_{B\alpha S\alpha}+x_{B\beta S\beta}+x_{B\gamma S\gamma} \geq u_{BoSo}. \quad \textbf{(27)}$$

Similarly, since $\omega_{BoS\alpha}=\omega_{B\alpha So}=\omega_{B\beta S\gamma}=\omega_{B\gamma S\beta}$, $\omega_{BoS\beta}=\omega_{B\alpha S\gamma}=\omega_{B\beta So}=\omega_{B\gamma S\alpha}$, and $\omega_{BoS\gamma}=\omega_{B\alpha S\beta}=\omega_{B\beta S\alpha}=\omega_{B\gamma So}$, the associated equations also can be rewritten. Hence, for the submatrix $\Omega_{\mathbf{B}\times\mathbf{S}}$, instead of 16 variables and 48 constraints, only 4 variables and 12 constraints are required.

The same process is also implemented to the other eight submatrices. The problem size is substantially reduced.

2. Eliminate Redundant Variables

Consider the loading pattern adjoined to the virtual layer V (x_{iV}); there is no difference among the four loading patterns of a layer type. Therefore, Eq. (8) can be reduced to

$$\sum_{j \in \mathbf{P}} x_{ij} + x_{iV} = y_i \ (\forall i \in \{Bo, So, Eo\}), \quad \textbf{(28)}$$

$$\sum_{j \in \mathbf{P}} x_{ij} = y_i \ (\forall i \in \mathbf{P}\text{-}\{Bo, So, Eo\}). \quad \textbf{(29)}$$

Nine x_{iV} variables are eliminated. Equation (11) can be rewritten as

$$x_{BoV}+x_{SoV}+x_{EoV}=1. \quad \textbf{(30)}$$

3. Bounds of the π Value

We provide three bounds of the π value to speed up the branch-and-bound solution procedure of Model O3 as follows.

Theorem IV.1.
$\omega^*_{\min}$ and ω^*_{ave} of Model O1 are the lower and upper bounds of π; that is, $\omega^*_{\min} \leq \pi^* \leq \omega^*_{\text{ave}}$, where $\omega^*_{\min}=\text{Min.}\{\omega^*_{p1p2}(1), \omega^*_{p2p3}(2), \ldots, \omega^*_{p(t-1)pt}(t-1)\}$.

Proof
Suppose $\pi' > \omega^*_{\text{ave}}$ is a feasible solution of Model O3. Then, there exists a stacking sequence $\hat{\omega}_{p1p2}(1), \ldots, \hat{\omega}_{p2p3}(2), \ldots, \hat{\omega}_{p(t-1)pt}(t-1)$ such that $\pi'=\text{Min}\{\hat{\omega}_{p1p2}(1), \hat{\omega}_{p2p3}(2), \ldots, \hat{\omega}_{p(t-1)pt}(t-1)\}$. So $\forall \hat{\omega}_{ij}(r) > \omega^*_{\text{ave}}$. Therefore,

$$\hat{\omega}_{p1p2}(1)+\hat{\omega}_{p2p3}(2)+\ldots+\hat{\omega}_{p(t-1)pt}(t-1) > (t-1)\omega^*_{\text{ave}} = \theta^* \quad \textbf{(31)}$$

Since Models O3 and O1 have the same constraints, the stacking $\hat{\omega}_{p1p2}(1), \hat{\omega}_{p2p3}(2), \ldots, \hat{\omega}_{p(t-1)pt}(t-1)$ is also feasible for Model O1, and $\hat{\omega}_{p1p2}(1)+\hat{\omega}_{p2p3}(2)+\ldots+\hat{\omega}_{p(t-1)pt}(t-1)$ should not be greater than θ^*. Therefore, Eq. (31) is in conflict to the definition of θ^* and ω^*_{ave} in Model O1. Therefore, any feasible solution of model O3, π, cannot be greater than ω^*_{ave}; that is, ω^*_{ave} should be the upper bound of Model O3.

Because Models O1 and O3 have the same constraints, $\omega^*_{\min}$ is also a feasible solution of Model O3. Therefore, $\omega^*_{\min}$ is a lower bound of π^*.

Theorem IV.2.
$\tilde{\pi}^*$ is the upper bound of π, $\pi^* \leq \tilde{\pi}^*$, where $\tilde{\pi}^*$ is the optimal solution of Model O3 without including Eq. (12).

Proof
Let the stacking sequence $\tilde{\omega}^*_{p1p2}(1)$, $\tilde{\omega}^*_{p2p3}(2)$, ..., $\tilde{\omega}^*_{p(t-1)pt}(t-1)$ have the optimal solution of Model O3-relax, $\tilde{\pi}^*$. If the stacking sequence does not violate Eq. (12), it is also the optimal solution of Model O3. Otherwise, the optimal solution of Model O3 will be less than $\tilde{\pi}^*$ because adding constraints to the maximum model would decrease the objective value.

Therefore, we need four steps to solve the max-min problem.

(1) Step 1: Solve Model O1 to obtain $\omega^*_{\min}$ and ω^*_{ave}. If $\omega^*_{\text{ave}} \leq 0$, then Model O3 has no feasible solution and go to *Adjustment Procedure*. Otherwise, go to Step 2.

(2) Step 2: Solve Model O3-relax with lower bound $\omega^*_{\min}$ and upper bound ω^*_{ave}. If $\tilde{\pi}^* < 0$, Model O3 has no feasible solution and go to *Adjustment Procedure*. If $\tilde{\pi}^* = \omega^*_{\min}$, the optimal solution of Model O3 is $\omega^*_{\min}$ and stop. If $\tilde{\pi}^* > \omega^*_{\min}$, go to Step 3.

(3) Step 3: Solve the Model O3 with lower bound $\omega^*_{\min}$ and upper bound $\tilde{\pi}^*$.

(4) Step 4: A trace of the solution of Model O3 from V to V forms a single non-simple path that is the stacking sequence of the unit load.

V. Adjustment Procedure

The five-phase method may be mathematically infeasible or may yield an impractical pallet loading design. Three cases might occur with this method. Case 1: $\omega^*_{\text{ave}} < 0$, there is no feasible solution for Model O3. Case 2: $\tilde{\pi}^* < 0$, there is no feasible solution for Model O3. Case 3: π^* is too small to stack for practical consideration.

Since the loading types generated in Phase 1 are the caused factors, we suggest three alternative procedures to increase the solution spaces. Method 1: Adjust the loading patterns generated in Phase 1 by using the *compacting, centering blocks and distributing gaps* procedures. Method 2: Find another basic loading pattern of B, S, or E-type layer in Phase 1. Method 3: Scarify some cube utilization by setting the objective of Phase 3 to less than Λ^*.

VI. A Numerical Example

The five-phase method was coded into a computer program running on a DOS-based PC. LINDO (Schrage, 1991) was used to solve the IP models of Phase 3 and Phase 5.

Consider the problem of loading a pallet with packages of potato chips packed in fiberboard boxes with l=37 cm, w=25 cm, h=20 cm, and m=2.3 kg. Assume that the size of the pallet is L=120 cm, W=100 cm, H=140 cm, and that the maximum weight is M=1500 kg. In this example, the value of the stability criterion is 10% for σ_s and 50% for σ_b. The effects of the Supportive criterion and Base contact criterion are equal, that is, $\rho_s = \rho_b = 0.5$.

Phase 1 reveals that the maximum number of boxes in a B-type layer is 12, in an S-type layer is 15, and in an E-type layer is 24, respectively. The area utilization of B-, S-, and E-type layers is 92.5%, 92.5% and 100.0%. Detailed results of Phase 1 are shown in Table 1. The twelve possible loading patterns formed in Phase 2 are shown in Fig. 2. In Phase 3, we can formulate the problem in the following way:

$$\Lambda^* = \text{Max.}(12Z_B + 15Z_S + 24Z_E),$$

subject to

$$20Z_B + 25Z_S + 37Z_E \leq 140,$$

$$12Z_B + 15Z_S + 24Z_E \leq 90,$$

where Z_B, Z_S, Z_E are integers.

Solving the above problem, the number of B-type layers (Z_B) is 2, of S-type layers (Z_S) is 1, and of E-type layers (Z_E) is 2. The stability coefficient matrix Ω obtained in Phase 4 is listed in Table 2. In Phase 5, we first solve Model O1, and the non-zero variables solution is $x_{B\gamma Eo}=2$, $x_{S\beta B\gamma}=1$, $x_{EoS\beta}=1$, $x_{EoV}=1$, $x_{VB\gamma}=1$.

Table 1. The Characteristics of B-, S-, and E-type Layers

	B-type	S-type	E-type
Surface area (A_t, cm×cm)	37(l)×25(w)	37(l)×20(h)	25(w)×20(h)
No. of boxes per layer (N_t)	12	15	24
Area Utilization (%)	92.5	92.5	100

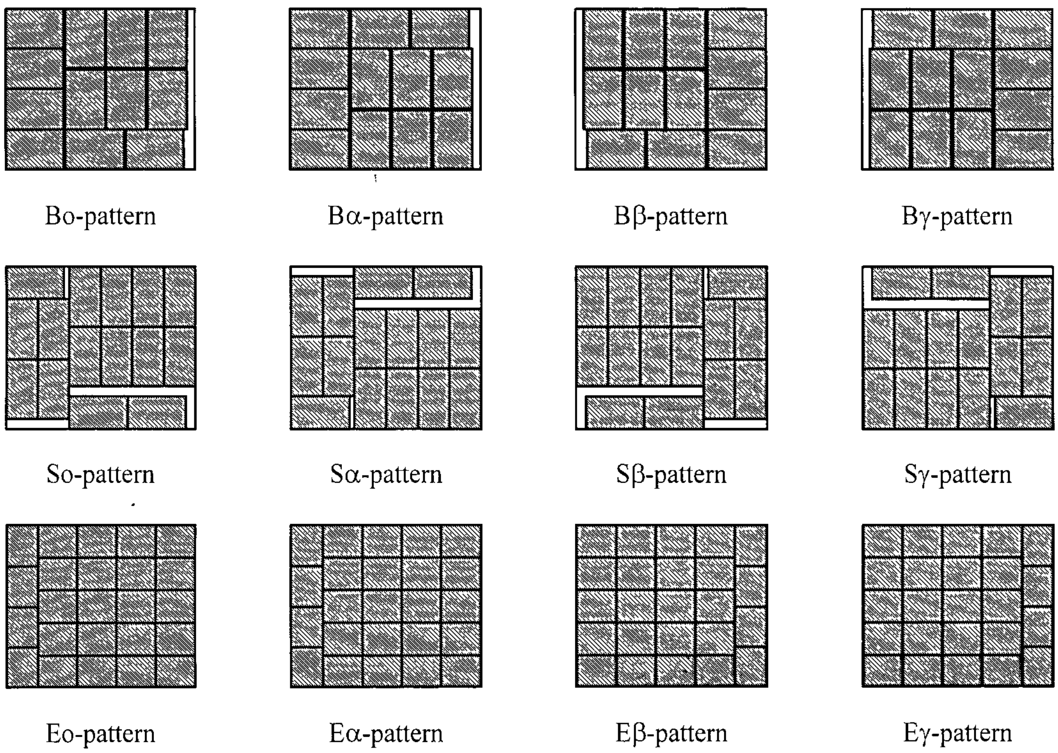

Fig. 2. Pallet stack construction.

Table 2. The Matrix of 144 Stability Coefficients for 12 Possible Patterns ($\Omega=[\omega_{ij}]$)

	Bo	*Bα*	*Bβ*	*Bγ*	*So*	*Sα*	*Sβ*	*Sγ*	*Eo*	*Eα*	*Eβ*	*Eγ*
Bo	0.63	0.70	0.69	0.71	0.69	0.79	0.76	0.76	0.91	0.91	0.96	0.96
Bα	0.70	0.63	0.71	0.69	0.79	0.69	0.76	0.76	0.91	0.91	0.96	0.96
Bβ	0.69	0.71	0.63	0.70	0.76	0.76	0.69	0.79	0.96	0.96	0.91	0.91
Bγ	0.71	0.69	0.70	0.63	0.76	0.76	0.79	0.69	0.96	0.96	0.91	0.91
So	0.64	0.75	0.71	0.69	0.62	0.71	0.69	0.65	0.82	0.82	0.81	0.81
Sα	0.75	0.64	0.69	0.71	0.71	0.62	0.65	0.69	0.82	0.82	0.81	0.81
Sβ	0.71	0.69	0.64	0.75	0.69	0.65	0.62	0.71	0.81	0.81	0.82	0.82
Sγ	0.69	0.71	0.75	0.64	0.65	0.69	0.71	0.62	0.81	0.81	0.82	0.82
Eo	0.67	0.67	0.67	0.67	0.69	0.69	0.69	0.69	0.62	0.62	0.77	0.77
Eα	0.67	0.67	0.67	0.67	0.69	0.69	0.69	0.69	0.62	0.62	0.77	0.77
Eβ	0.67	0.67	0.67	0.67	0.69	0.69	0.69	0.69	0.77	0.77	0.62	0.62
Eγ	0.67	0.67	0.67	0.67	0.69	0.69	0.69	0.69	0.77	0.77	0.62	0.62

Since the total stability coefficient (θ^*) is 3.36, we set the upper and lower bounds of Model O3-relax to $\omega^*_{\text{ave}}=3.36/4=0.84$ and $\omega^*_{\text{min}}=0.69$. The solution of Model O3-relax, 0.75, is the upper bound of Model O3. After running Model O3, the non-zero variables solution is $x_{B\alpha So}=1$, $x_{SoB\alpha}=1$, $x_{B\alpha E\beta}=1$, $x_{E\beta Eo}=1$, $x_{EoV}=1$, $x_{VB\alpha}=1$. The trace of the non-zero variables solution is $x_{VB\alpha} \rightarrow x_{B\alpha So} \rightarrow x_{So\beta\alpha} \rightarrow x_{B\alpha E\beta} \rightarrow x_{E\beta Eo} \rightarrow x_{EoV}$. The stacking sequences of the unit load from the top to the pallet are *Bα*, *So*, *Bα*, *Eβ*, and *Eo* as shown in Fig. 3.

VII. Performance Analysis

To evaluate the performance of the proposed method, box sizes were generated in 2 cm increments, with h=10 cm to 20 cm, $w-h$=0 cm to 10 cm, $l-w$=0 cm to 10 cm, for a total of 216 (=6×6×6) box sizes. The three pallet sizes, 110×110 cm, 120×100 cm, and 120×80 cm (Jansen, 1983), were set to the pallet specifications (L, W). The loading height and weight were H=140 cm, and M=1500 kg.

The weighting factors ρ_s and ρ_b influenced the

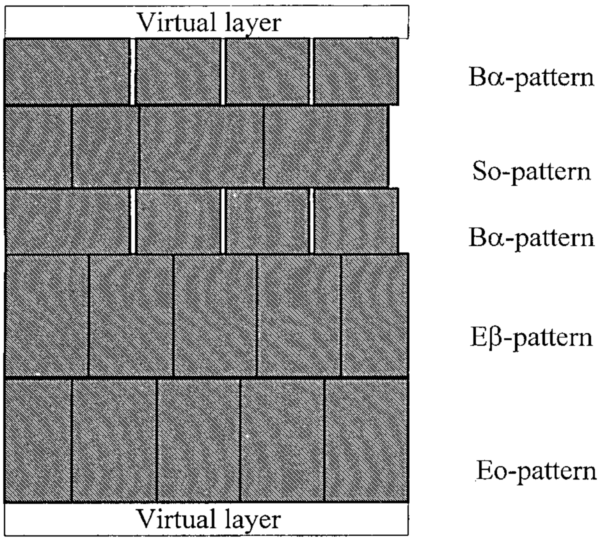

Fig. 3. Stacking sequence.

stability coefficients. Five combinations of weighting factors were set for each box and pallet combination to test the sensitivity. Each setting was solved for the max-sum and max-min stability models.

Table 4 shows the results for the 110×110 cm pallet size. For instance, with a combination of 0.25 and 0.75 for the 216 box sizes, two box sizes no feasible solutions that need to be modified by the *Adjustment Procedure.* There are 35 box sizes that have $\pi^* > \omega^*_{\min}$. The average improvement for these 35 box sizes is 1.77%. The maximum improvement among these 35 box sizes is 5.06%. The number of improvement solutions decreases as the ρ_s value increases. The average and maximum improved values ($\pi^* - \omega^*_{\min}$) increase as the ρ_s value increases.

From Eq. (1), the possible values for CN_k/CN^* are: 1/4, 2/4, 3/4, and 4/4. The value of CA_k/CA^* is in the range [0.5, 1]. By definition, the CA_k/CA^* values are only slightly different. Hence, the variance of the elements in the matrix Ω becomes more sensitive as the ρ_s value increases.

If the Base contact criterion is more significant in defining the stability coefficient, as ρ_s=0 and ρ_b=1.0, the max-min model has more improved solutions but with little improvement. On the other hand, if the Supportive criterion is more significant, as the ρ_s=1 and

Table 3. Stacking Sequence of Unit Load for Model O1, O3-relax, and O3

Model	Stacking Sequence	Stability Coefficients	Total	Max.	Min.
O1	*Bγ, Eo, Sβ, Bγ, Eo*	0.96, 0.69, 0.75, 0.96	3.36	0.96	0.69
O3-relax	*(V,Bo), (Bα,So), (Eo,Eβ)*	0.79, 0.75, 0.77, 0.77	3.08#	0.79#	0.75#
O3	*Bα, So, Bα, Eβ, Eo*	0.79, 0.75, 0.96, 0.77	3.27	0.96	0.75

Notes: ( ): a subtour
#: it is not a feasible solution of Model O3

Table 4. Computational Results for 110×110 cm Pallet

ρ_s		0	0.25	0.5	0.75	1.0
ρ_b		1.0	0.75	0.5	0.25	0
No. of no feasible solution		2	2	2	2	2
No. of feasible solutions has $\pi^* > \omega^*_{\min}$	(n_t)	55	35	27	23	24
$\Sigma[(\pi^* - \omega^*_{\min})/\omega^*_{\min}]/n_t$	(%)	1.49	1.77	4.11	8.29	12.12
Max.$[(\pi^* - \omega^*_{\min})/\omega^*_{\min}]$	(%)	6.67	5.06	19.05	26.92	50.00

Table 5. Computational Results for 120×100 cm Pallet

ρ_s		0	0.25	0.5	0.75	1.0
ρ_b		1.0	0.75	0.5	0.25	0
No. of no feasible solution		5	5	5	5	5
No. of feasible solutions has $\pi^* > \omega^*_{\min}$	(n_t)	67	34	38	30	29
$\Sigma[(\pi^* - \omega^*_{\min})/\omega^*_{\min}]/n_t$	(%)	1.29	1.50	3.97	6.82	9.77
Max.$[(\pi^* - \omega^*_{\min})/\omega^*_{\min}]$	(%)	5.62	8.33	23.53	33.33	32.61

Table 6. Computational Results for 120×80 cm Pallet

ρ_s		0	0.25	0.5	0.75	1.0
ρ_b		1.0	0.75	0.5	0.25	0
No. of no feasible solution		4	4	4	4	4
No. of feasible solutions has $\pi^*>\omega^*_{\min}$	(n_t)	26	36	20	18	25
$\Sigma[(\pi^*-\omega^*_{\min})/\omega^*_{\min}]/n_t$	(%)	1.53	2.39	5.05	10.39	13.24
Max $[(\pi^*-\omega^*_{\min})/\omega^*_{\min}]$	(%)	4.26	10.53	12.68	22.22	45.71

Table 7. Average and Variance of π^*

ρ_s	0		0.25		0.5		0.75		1.0	
ρ_b	1.0		0.75		0.5		0.25		0	
	average	variance	average	variance	average	variance	average	variance	average	variance
110×110 cm	0.98	0.0009	0.85	0.0013	0.73	0.0047	0.61	0.0088	0.49	0.0159
120×100 cm	0.99	0.0003	0.85	0.0010	0.72	0.0042	0.59	0.0090	0.46	0.0168
120×80 cm	0.99	0.0006	0.85	0.0012	0.71	0.0069	0.59	0.0093	0.47	0.0171

ρ_b=0, the max-min model has fewer improved solutions, but the solutions have more improvement.

For these products that mainly depend on the interlock of the boxes for loading stability, the effect of the Supportive criterion is more significant than that of the Base contact criterion. The max-min model will provide better performance for the pallet loading design.

Tables 5 and 6 display the results for the 216 box sizes loaded on pallets with size 120×100 cm and 120×80 cm, respectively. The observations given in the previous paragraph are also relevant here.

Table 7 displays the averages and variances of the π^* value of the 216 box sizes for each pallet under five weighting combinations. On average, the π^* value decreases as the ρ_s value increases. Under each weighting combination, the average of the π^* values for the three pallets are similar. We may conclude that the performance of the proposed method has no significant difference for the pallet size.

VIII. Conclusions

In this paper, a five-phase method with an adjustment procedure has been proposed to solve the three-dimensional pallet loading problem. Under the first objective, greatest cube utilization, the method is to search for loading patterns of the stacking sequence such that the smallest value of the stability coefficient between adjoining layers is as high as possible. Potential applications of this method are numerous since many products can be positioned and transported on their end or side surfaces without incurring damage, and the stability of unit load often is influenced by the interface with the smallest value of the stability coefficient.

The computational time for one pallet with 216 box sizes is about 6 hours on a Pentium-100 PC. The average computational time for single-size boxes to be stacked is about 100 seconds, which is acceptable for practical design purposes. However, one box and pallet combination may have many different loading patterns. The greater the number of loading patterns that can be selected, the more the objective function of maximum stability can be improved. The complexity of the problem increases with the number of different loading patterns.

The heuristics of seeking the best sequence in Phase 5 can be stated in terms of graph theory. The loading sequence problem can be treated as a Transportation Problem or a Traveling Salesman Problem with side conditions. A variety of methods can be applied to solve the problem. It will be worthwhile to study these methods.

This method can be extended to cases where the boxes are not all the same size. If two-dimensional pallet loading with mixed box sizes is known in Phase 1, then the max-min stability problem of the stacking sequence of the unit load can be obtained by Phase 2 through Phase 5.

References

Bischoff, E. and W. B. Dowsland (1982) An application of the micro to product design and distribution. *J. Opl. Res. Soc.*, **33**, 271-281.

Bischoff, E. E. (1991) Stability aspects of pallet loading. *OR Spektrum*, **13**, 189-197.

Carpenter, H. and W. B. Dowsland (1985) Practical considerations

of the pallet loading problem. *J. Opl. Res. Soc.*, **36**, 489-497.
Cox, R. M. and K. G. Van Tassel (1985) The role of packaging in physical distribution. In: *The Distribution Handbook*, pp. 737-773. The Free Press, New York, NY, U.S.A.
Liu, F. H. and C. J. Hsiao (1997) A three-dimensional pallet loading method for single-size boxes. *J. Opl. Res. Soc.*, **48**, 726-735.
Jansen, R. (1983) Standardization of packing and load units in the European area. *Material Flow*, **1**, 157-163.
Schrage, L. (1991) *User's Manual for Linear, Integer, and Quadratic Programming with LINDO*. The Scientific Press, South San Francisco, CA, U.S.A.
Smith, A. and P. de Cani (1980) An algorithm to optimize the layout of boxes in pallets. *J. Opl. Res. Soc.*, **31**, 573-578.
Steudel, H. J. (1979) Generating pallet loading patterns: a special case of the two-dimensional cutting stock problem. *Mgmt. Sci.*, **25**, 997-1004.

單一尺寸箱體棧板堆疊方法—最佳化堆疊穩定度

劉復華　蕭智仁

國立交通大學工業工程與管理學系

摘　要

箱體堆疊於棧板上使其箱體數最大的問題稱爲箱體疊棧問題。如果箱體能以底面、側面或端面堆疊，棧板的體積使用率會增加，但單元負載的穩定度可能降低。爲了量化單元負載的穩定度，本論文定義兩堆疊層之接觸面的穩定係數。根據穩定係數，本論文提出一單一尺寸箱體疊棧方法，使最大裝載箱數的單元負載能有最大的穩定度。本論文以一實例說明，並利用216種箱體與三種標準棧板的組合做測試與比較。