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Statistically Minimum-loss Design of Averages Control Charts for Non-normal Data

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ABSTRACT

When an $\bar{x}$ control chart is used to monitor a process, three parameters should be determined: the sample size, the sampling interval between successive samples, and the control limits of the chart. Duncan presented a cost model to determine the three parameters for an $\bar{x}$ chart. In their 1995's paper, S.M. Alexander and coworkers combined Duncan's cost model with the Taguchi loss function to present a loss model for determining the three parameters. In this paper, the Burr distribution is employed to obtain a statistical minimum-loss design of $\bar{x}$ charts for non-normal data. The Alexander's loss model is used as the objective function, and the cumulative function of the Burr distribution is applied to derive the statistical constraints of the design. An example is presented to illustrate the solution procedure. From the results of the sensitivity analyses, we find that small values of the skewness coefficient (say, $\alpha_3 < 0.4$) have no significant effect on the optimal design; however, when $\alpha_3 > 0.4$, an increase in α_3 leads to slight increases in both the sample size and the sampling interval, and to a wider control limit. Meanwhile, an increase in the kurtosis coefficient (α_4) results in an increase in the sample size and in a wider control limit.

Key Words: loss function, control chart, statistically minimum-loss design, the Burr distribution

I. Introduction

Statistical methods have been widely applied in industrial process control. One of the key methods is the control chart technique, which may be considered as a graphical expression of statistical hypothesis testing. Since 1924 when Dr. W.A. Shewhart presented the first control chart, various control chart techniques have been developed and applied in process control. The major function of control charting is to detect the occurrence of assignable causes so that the necessary corrective action may be taken before a large number of nonconforming products are manufactured. A survey by Saniga and Shirland (1977) indicated that the averages control chart (also called the $\bar{x}$ chart) is used more often than any other control chart techniques when quality is measured on a continuous scale. When an $\bar{x}$ control chart is used to monitor a process, three parameters should be determined: the sample size, the sampling interval between successive samples, and the control limits or critical region for the chart. Shewhart (1939) suggested 3-sigma control limits as action limits and sample sizes of four or five, leaving the interval between successive samples to be determined by the practitioner.

Duncan (1956) proposed the first model for determining the sample size (n), the interval between successive samples (h), and the control limits of an $\bar{x}$ control chart (i.e., $\mu \pm k\sigma/\sqrt{n}$, where μ and σ are, respectively, the mean and standard deviation of the process characteristic) which minimizes the average cost when a single out-of-control state (assignable cause) exists. Duncan's cost model includes the cost of sampling and inspection, the cost of defective products, the cost of false alarms, the cost of searching for assignable causes, and the cost of process correction. Since then, considerable attention has been devoted to the optimal economic determination of these three parameters (Duncan, 1971; Gibra, 1971; Goel *et al.*, 1968; Knappenberger and Grandage, 1969). Reviews of the literature in economic designs of control charts have been published by Montgomery (1980), Vance (1983) and Ho and Case (1994). Alexander *et al.* (1995) combined Duncan's cost model with the Taguchi loss function to obtain a loss model for determining the three parameters. This loss model explicitly considers the quality loss due to process variability, which is not included in Duncan's cost model.

In addition to focusing on economic design, another approach to designing a control chart is called statistical design. Statistically designed control charts are those in which control limits (which determine the Type I error probability, α) and power are preselected. These then determine the sample size and, if the average time to signal is specified, the sampling interval (Woodall, 1985). McWilliams (1994) incorporated the concept of statistical considerations into the economic design of control charts and then presented the "economic statistical design" of $\bar{x}$ control charts for normal data. An economic statistical design for joint $\bar{x}$ and R charts has also been proposed by Saniga (1989).

Traditionally, when designing control charts, one usually assumes that the measurements in the sample are normally distributed. However, this assumption may not be tenable. If the measurements really are normally distributed, then the statistic $\bar{x}$ is also normally distributed. If the measurements are asymmetrically distributed, then the statistic $\bar{x}$ will be approximately normally distributed only when the sample size n is sufficiently large (based on the central limit theorem). Unfortunately, when a control chart is applied to monitor the process, the sample size n is never sufficiently large due to the sampling cost. Therefore, if the measurements are not normally distributed, the traditional way of designing a control chart may reduce the ability of the control chart to detect the assignable causes. Yourstone and Zimmer (1992) used the Burr distribution to represent various non-normal distributions and, consequently, to statistically design the control limits of an $\bar{x}$ control chart. However, they did not consider cost in the design of the chart.

In this paper, the statistical minimum-loss design of an $\bar{x}$ control chart for non-normal measurements will be developed using the Burr distribution. In the next section, Duncan's cost model (Duncan, 1956) and the loss model given by Alexander *et al*. (1995) will be briefly reviewed. Alexander's loss model will be employed as the objective function, which is to be minimized. Then, the Burr distribution will be applied to represent various non-normal distributions. Based on the cumulative function of the Burr distribution, the calculations of the α risk and the power of the chart will be derived, and the upper bound of the α risk and the lower bound of the power will be specified as statistical constraints of the design. An example will be presented to illustrate the solution procedure for the statistical minimum-loss design of an $\bar{x}$ control chart under non-normality. Some sensitivity analyses will be conducted to evaluate the effects of model parameters, non-normality, and statistical constraints on the statistical minimum-loss design of an $\bar{x}$ control chart.

II. Literature Reviews

In this section, we shall briefly review the cost and loss models given by Duncan (1956) and Alexander *et al*. (1995) and the cumulative function of the Burr distribution, which is employed to represent various non-normal distributions in this paper.

1. Review of the Cost and Loss Models

According to Alexander *et al*. (1995), Duncan's cost model for $\bar{x}$ control charts is more realistic than other models. The components of Duncan's cost model include:

(1) the cost of an out-of-control condition,
(2) the cost of false alarms,
(3) the cost of finding an assignable cause, and
(4) the cost of sampling, inspection, evaluation, and plotting.

Duncan (1956) assumed that the process starts under an in-control condition and is subject to random shifts in the process mean. Once a shift occurs, the process remains there until it is corrected. The cycle length is defined as the total period from when the process starts in-control, to when it shifts to an out-of-control condition, to when the out-of-control condition is detected, which results in the assignable cause being identified. These four time intervals are, respectively, the interval during which the process is in-control, the interval during which the process is out-of-control before the final sample of the detecting subgroup is taken, the interval used to sample, inspect, evaluate and plot the subgroup results, and the interval used to search for the assignable cause. When the average cycle length is determined, the cost components can be converted to a "per hour of operation" basis. Given associated cost and time parameters, the optimal values of the three decision parameters for the model can then be determined by using optimization techniques.

In Duncan's model (Duncan, 1956), the four average cycle length components are as follows:

(1) Assuming that the process begins in the in-control state, the time interval during which the process remains in control is an exponential random variable with mean $1/\lambda$, which is the average process in-control time.
(2) When an assignable cause occurs, the probability that this out-of-control condition will be detected on any subsequent sample is $1 - \beta$, which is the power of the chart. Thus, the expected number of subgroups taken before a shift in the process mean is detected is $1/(1 - \beta)$. The average time of occurrence within an interval between the jth and $(j +$

1)st subgroups, given an occurrence of a shift in the interval between these subgroups, is

$$\tau = \frac{1-(1+\lambda h)e^{-\lambda h}}{\lambda(1-e^{-\lambda h})}.$$

Therefore, the expected length of the out-of-control period is $h/(1-\beta)-\tau$.

(3) The average sampling, inspecting, evaluating, and plotting time for each sample is a constant g proportional to the sample size n, so that the delay in plotting a subgroup point on the $\bar{x}$ chart is gn.

(4) The time needed to find the assignable cause following an action signal is a constant D.

Therefore, the expected length of a cycle, denoted by $E(T)$, is

$$E(T) = \frac{1}{\lambda} + \frac{h}{1-\beta} - \tau + gn + D, \qquad \textbf{(1)}$$

and the expected cost per hour, denoted by $E(C)$, incurred by the process is

$$E(C) = \frac{a_1 + a_2 n}{h} + \frac{a_4[E(T)-(1/\lambda)] + a_3 + a a_5 e^{-\lambda h}/(1-e^{-\lambda h})}{E(T)}, \qquad \textbf{(2)}$$

where a_1 and a_2 are, respectively, the fixed and variable components of the sampling cost, a_3 is the cost of finding an assignable cause, a_4 represents the hourly penalty cost associated with production in the out-of-control state, and a_5 is the cost of investigating a false alarm. The economic design of an $\bar{x}$ chart involves determining the appropriate values of n, h and k such that $E(C)$ can be minimized.

Taguchi and Wu (1985) defined product quality as the loss a product imparts to society from the time the product is shipped; consequently, they introduced the quality loss function as a quality performance measure for a product. Consider a product with bilateral tolerances of equal value (Δ). If the loss (or cost) to society of producing a product out of specification is A \$/unit, then the Taguchi loss function defines the expected loss to society as

$$\text{Expected loss per unit} = \frac{A}{\Delta^2} v^2, \qquad \textbf{(3)}$$

where v^2 is the mean squared deviation of the process, defined as $v^2 = \sigma^2 + (\mu - T)^2$, and T is the target of the process characteristic. When the process is in control, its mean is assumed to be centered on the target (i.e., $\mu = T$) and $v^2 = v_1^2 = \sigma^2$. When the process mean shifts to $\mu = T + \delta\sigma$, $v^2 = v_2^2 = \sigma^2(1 + \delta^2)$. Using the definition of loss given in Eq. (3), Alexander *et al*. (1995) expanded Duncan's cost model to consider losses due to in-control and out-of-control variability. By assuming that the production rate is P units/hr and applying some approximations on the terms of Eq. (2), the expected cost (or loss) per hour, denoted by $E(L)$, can be obtained as

$$E(L) = \frac{a_1 + a_2 n}{h} + \frac{a_3\lambda + a_5\alpha/h + L_1 P + L_2 P\lambda B}{1+\lambda B}, \qquad \textbf{(4)}$$

where $B = [1/(1-\beta) - 1/2 + \lambda h/12]h + D + gn$, and $L_i = (A/\Delta^2)v_i^2$, for $i = 1$ and 2. The minimum-loss design of an $\bar{x}$ chart involves determining the optimal values of n, h and k such that $E(L)$ is minimized.

2. Review of the Burr Distribution

Rahim (1985) presented an economic model of an $\bar{x}$ control chart under non-normality but did not include statistical consideration in the design. Meanwhile, Rahim's approach to transforming the standardized normal variate to non-normal variates is more complicated than the Burr-distribution approach presented by Yourstone and Zimmer (1992). In this paper, we shall use the Burr distribution to represent various non-normal distributions. The cumulative distribution function of the Burr distribution (Burr, 1942) has the following concise form:

$$F(y) = 1 - \frac{1}{(1+y^c)^q}, \quad \text{for } y \geq 0, \qquad \textbf{(5)}$$

where c and q are greater than one. By taking the first derivative on Eq. (5), the density function of the Burr distribution can be obtained as

$$f(y) = \frac{qcy^{c-1}}{(1+y^c)^{q+1}}, \quad \text{for } y \geq 0.$$

Different combinations of c and q cover a wide range of skewness and kurtosis coefficients of various probability density functions, including most of the known functions, such as normal, Gamma, Beta, and so forth. For example, the normal density function can be approximated by the Burr distribution with c = 4.85437 and q = 6.22665. Application of the Burr distribution can be found in the literature. Burr (1967) applied the Burr distribution to study the effect of non-normality on the constants of $\bar{x}$ and range control charts. Zimmer and Burr (1963) used the Burr distribution to develop variable sampling plans for non-normal populations. Yourstone and Zimmer (1992) used the Burr distribution to design the control limits of an $\bar{x}$ control charts for non-normal data. Chou and Cheng

(1997) extended Yourstone and Zimmer's model to design the control limits of a range control chart under non-normality. Also, Tsai (1990) employed the Burr distribution to design the probabilistic tolerance for a subsystem.

Burr (1942) tabulated the expected value, standard deviation, skewness coefficient and kurtosis coefficient of the Burr distribution for various combinations of c and q. These tables allow users to make a standardized transformation between a Burr variate (say, Y) and another random variate (X). For a given set of data, once the sample skewness and kurtosis coefficients are estimated, the tables given by Burr (1942) can be used to obtain the mean and standard deviation of the corresponding Burr distribution. For example, if a set of data is collected and the corresponding sample statistics are computed to obtain a sample mean ($\bar{x}$) of 50.42, a sample standard deviation (s_x) of 5.68, a sample skewness coefficient ($\hat{\alpha}_3$) of 0.18, and a sample kurtosis coefficient ($\hat{\alpha}_4$) of 3.05, then from Table III in Burr (1942), this set of data can be approximately described by a Burr distribution with $c = 4$ and $q = 6$. Let M and S be the mean and standard deviation of a Burr random variate, respectively. From Table II in Burr (1942), the Burr random variate Y with $c = 4$ and $q = 6$ has a mean $M = 0.5951$ and a standard deviation $S = 0.1801$. Then, the standardized transformation between a Burr variate (Y) and the random variate (X) of interest can be expressed as follows:

$$\frac{X-\bar{x}}{s_x} = \frac{Y-M}{S}$$

$$\rightarrow \frac{X-50.42}{5.68} = \frac{Y-0.5951}{0.1801}$$

$$\rightarrow X = 31.652 + 31.538Y.$$

III. Statistical Constraints

To derive the statistical constraints for the design of an $\bar{x}$ control charts, we first denote UCL and LCL as the upper and lower control limits of the $\bar{x}$ chart, respectively. In mathematical expression,

$$UCL = T + k\frac{\sigma}{\sqrt{n}},$$

$$LCL = T - k\frac{\sigma}{\sqrt{n}}. \tag{6}$$

Note that we assume that $\mu = T$ when the process is in the in-control state. The Burr random variate Y can be transformed into the sample statistic $\bar{x}$ by using the standardized procedure as follows:

$$\frac{Y-M}{S} = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}}. \tag{7}$$

That is, the scale and origin of the fitted Y values can be changed to those of the $\bar{x}$ values, and from Eq. (7), when the process is in-control, we have

$$\bar{x} = T + (Y-M)\frac{\sigma/\sqrt{n}}{S}. \tag{8}$$

Based on Eqs. (5), (6) and (8), the Type I error probability of the $\bar{x}$ chart is

$$\begin{aligned}\alpha &= P(\bar{x} > UCL) + P(\bar{x} < LCL)\\ &= P(Y > M + kS) + P(Y < M - kS)\\ &= 1 + \frac{1}{[1+(M+kS)^c]^q} - \frac{1}{[1+(M-kS)^c]^q}.\end{aligned} \tag{9}$$

When $\mu = T + \delta\sigma$ (i.e., the process mean has shifted), $\bar{x}$ is assumed to follow a Burr distribution with mean $T + \delta\sigma$ and standard deviation $\sigma/\sqrt{n}$. The power of an $\bar{x}$ chart, denoted by p, is equal to $1 - \beta$, where β is the Type II error probability and can be expressed as

$$\beta = P(LCL \le \bar{x} \le UCL \mid \mu = T + \delta\sigma). \tag{10}$$

In order to compute the Type II error probability, the standardized transformation procedure has to be used as follows:

$$\frac{Y-M}{S} = \frac{\bar{x}-(T+\delta\sigma)}{\sigma/\sqrt{n}}$$

or

$$\bar{x} = T + \delta\sigma + (Y-M)\frac{\sigma/\sqrt{n}}{S}. \tag{11}$$

Based on Eqs. (5), (6) and (11), the Type II error probability of the $\bar{x}$ chart can be computed as

$$\begin{aligned}\beta &= P(LCL \le \bar{x} \le UCL \mid \mu = T + \delta\sigma)\\ &= P(M - kS - S\delta\sqrt{n} \le Y \le M + kS - S\delta\sqrt{n})\\ &= \frac{1}{[1+(M-kS-S\delta\sqrt{n})^c]^q}\\ &\quad - \frac{1}{[1+(M+kS-S\delta\sqrt{n})^c]^q}.\end{aligned} \tag{12}$$

Equations (9) and (12) represent computation of the Types

I and II error probabilities of the $\bar{x}$ chart. When statistical consideration is included in the design of an $\bar{x}$ chart, the upper bound of the Type I error probability (α_u) and the lower bound of the power (p_l) are specified such that $\alpha \le \alpha_u$ and $p = 1 - \beta \ge p_l$.

IV. An Example and Its Solution

In this section, an example is presented to illustrate the solution procedure of the economical statistical design of an $\bar{x}$ chart. The model parameters used in this example are borrowed directly from Alexander *et al*. (1995) so that the solution can be compared with the solution obtained based on the assumption normality.

Example. A plant manufactures packed orange juice that has a "quantity of content" specification of 250 cc with a tolerance of ±0.3 cc. From past data, the process standard deviation is estimated as 0.1 cc. Process shifts occur at random with a frequency of about one every 4 hours of operation ($\lambda = 0.25$). The manufacturer uses an $\bar{x}$ chart to monitor the process. Based on an analysis of quality-control technicians salaries and the costs of test equipment, it is estimated that the fixed cost of taking a sample is \$1 (i.e., $a_1 = 1$). The estimated variable cost of sampling is about \$0.10 per quantity of content (i.e., $a_2 = 0.10$) and it takes approximately 0.01 hour (i.e., $g = 0.01$) to measure and record the quantity of content of a bottle of orange juice. On average, when the process goes out of control, the magnitude of the shift is approximately one standard deviation ($\delta = 1.0$). The average time required to investigate an out-of-control signal is two hours (i.e., $D = 2$). The cost of investigating an action signal that results in the elimination of an assignable cause is \$50 while the cost of investigating a false alarm is \$50 (i.e., $a_3 = 50$ and $a_5 = 50$). The process is assumed to continue to produce packed orange juices at a rate of 100/h during the period of investigation and elimination of out-of-control signals (i.e., $P = 100$). The cost of reworking or scrapping a package of juice that is found to be outside the specification limits is \$5 (i.e., $A = 5$). Previous data indicate that the skewness and kurtosis coefficients of the quantity of content of the packed orange juice are approximately 0.4836 and 3.3801, respectively, which may be described by a Burr distribution with $c = 3$ and $q = 6$. The manufacturer wishes to design the minimum-loss $\bar{x}$ chart statistically such that the statistical properties of the chart can be maintained and the costs (or losses) can be reduced. To satisfy the statistical requirements, the $\bar{x}$ chart would be designed with a Type I error probability (α) of less than 0.005 and a power (p) greater than 0.9.

A SIMSCRIPT computer program was coded for minimization of the loss model in Eq. (4) subject to the statistical constraints $\alpha_u = 0.005$ and $p_l = 0.9$. The program uses the grid-search approach to find the optimal values of n, h and k by evaluating a wide range of possible solutions. For a certain combination of n, h and k, the program calculates the corresponding α risk and power so as to examine whether or not this combination is a feasible solution (i.e., whether the combination satisfies the statistical requirements). The output from this program, using the values of the model parameters given in the example, is shown in Table 1. The program calculates the optimal control limit width k and sampling frequency h for various values of n, and the resulting value of the cost function in Eq. (4). The optimal control chart design can be found by inspecting the values of the cost function so as to find the minimum. From Table 1, no feasible solution can be obtained for $n = 1, 2, \ldots, 18$. We also note that the minimum cost is \$88.78 per hour, that the optimal $\bar{x}$ chart uses samples of size $n = 19$, that the control limits are located at $\pm k\sigma$, with $k = 3.03$, and that the samples are taken at intervals of $h = 1.15$ hour (approximately once every 69 minutes). The Type I error probability of this design is $\alpha = 0.005$, and the power of the chart is $p = 0.91886$.

V. Sensitivity Analysis

In this section, we shall study the effects of model parameters, non-normality and statistical constraints on the solution of the above-mentioned example.

Table 2 shows the effects of model parameters on the statistical minimum-loss design of the $\bar{x}$ chart. Increasing the fixed cost of sampling (a_1) increases the interval between samples (h) and the average cost. Large values of the variable cost of sampling (a_2) and the cost of finding an assignable cause (a_3) imply relatively infrequent sampling and average cost. The cost of investigating a false alarm (a_5) is relatively robust to the optimal design. Large values of the cost of reworking an out-of-specification product (A) lead to more frequent sampling. Increasing the average process in-control time (i.e., de-

Table 1. Output Solutions for the Presented Example

n	H	k	α risk	Power	Cost
19	1.15	3.03	0.00500	0.918860	88.78
20	1.19	3.03	0.00500	0.939008	88.80
21	1.23	3.03	0.00500	0.955365	88.84
22	1.26	3.03	0.00500	0.968362	88.89
23	1.30	3.03	0.00500	0.978435	88.96
24	1.33	3.03	0.00500	0.986008	89.03
25	1.35	3.03	0.00500	0.991489	89.12
26	1.37	3.08	0.00455	0.993538	89.21
27	1.39	3.14	0.00406	0.994851	89.30
28	1.41	3.20	0.00362	0.995924	89.39
29	1.43	3.26	0.00322	0.996797	89.48
30	1.45	3.33	0.00282	0.997296	89.57

Table 2. Effects of Model Parameters on the Optimal Design of the $\bar{x}$ Chart

		n	h	k	α risk	Power	Cost
a_1	0.1	19	0.96	3.03	0.004999	0.918860	87.92409
	1.0	19	1.15	3.03	0.004999	0.918860	88.77787
	10	23	2.62	3.03	0.004999	0.978435	93.60630
a_2	0.01	21	0.78	3.03	0.004999	0.955365	86.95367
	0.1	19	1.15	3.03	0.004999	0.918860	88.77787
	10	19	29.991	3.03	0.004999	0.918860	113.43511
a_3	25	19	1.07	3.03	0.004999	0.918860	85.13835
	50.0	19	1.15	3.03	0.004999	0.918860	88.77787
	100	19	1.39	3.03	0.004999	0.918860	95.95809
a_5	25	19	1.14	3.03	0.004999	0.918860	88.71448
	50	19	1.15	3.03	0.004999	0.918860	88.77787
	100	19	1.18	3.03	0.004999	0.918860	88.90183
A	0.5	19	29.991	3.12	0.004212	0.900335	11.85082
	5	19	1.15	3.03	0.004999	0.918860	88.77787
	50	19	0.31	3.03	0.004999	0.918860	780.19787
λ	0.025	20	2.21	3.03	0.004999	0.939008	62.58911
	0.25	19	1.15	3.03	0.004999	0.918860	88.77787
	2.5	19	29.991	3.12	0.004212	0.900335	111.34118
δ	0.7	38	1.53	3.03	0.004999	0.910123	90.88935
	1.0	19	1.15	3.03	0.004999	0.918860	88.77787
	10.0	2	0.73	9.99	0.000000	1.000000	85.82883
g	0.001	20	1.16	3.03	0.004999	0.939008	88.13406
	0.01	19	1.15	3.03	0.004999	0.918860	88.77787
	0.1	19	1.45	3.03	0.004999	0.918860	93.61177
D	0.2	19	0.84	3.03	0.004999	0.918860	79.65102
	2.0	19	1.15	3.03	0.004999	0.918860	88.77787
	20.0	19	4.04	3.03	0.004999	0.918860	105.43743
P	25.0	19	10.86	3.03	0.004999	0.918860	27.68145
	100.0	19	1.15	3.03	0.004999	0.918860	88.77787
	200.0	19	0.74	3.03	0.004999	0.918860	166.91139
Δ	0.0003	19	0.1	3.03	0.004999	0.918860	7593.60923
	0.003	19	1.15	3.03	0.004999	0.918860	88.77787
	0.3	19	29.991	3.12	0.004212	0.900335	1.26419

creasing the value of λ) produces a lower cost. The magnitude of the process mean shift (δ) has a significant effect on the design. A larger value of δ leads to a smaller sample size and a short sampling interval. The average sampling, inspecting, evaluating and plotting time for each sample (g) has no significant effect on the design. The time required to find the assignable cause (D) affects the sampling interval (h). Large values of D correspond to infrequent sampling. A larger value of the production rate (P) leads to a short sampling interval. Looser tolerances result in infrequent sampling, which is consistent with our intuitive reasoning.

Table 3 lists the optimal designs of the $\bar{x}$ chart in the example presented above for various combinations of the skewness coefficient (α_3) and kurtosis coefficient (α_4) of the population. To study the effect of non-normality on the optimal design of the $\bar{x}$ chart, the possible populations are divided into six groups in Table 3. In Groups I and III, when the value of α_4 is approximately fixed and α_3 increases from a negative to a positive value, no specific tendency of n, h and k can be observed. In Group II, when α_3 is close to zero and α_4 increases, the sample size increases slightly, and the control limits become wider. In Group IV, when $\alpha_4 > 4$ and the value of α_3 increases from 0.4 to 1.0, both the sample size and sampling interval increase slightly, and the control limits become wider. In Group V, when α_3 is close to one and α_4 increases from 4.4 to 7.2, the control limits become wider. In Group VI, when α_3 increases from 1.06 to 3.18 and α_4 increases from 7.2 to 38, both the sample size and the sampling interval generally increase, and the control limits become wider. Based on the observations from Table 3, we may draw the following conclusions:

(1) Small values of the skewness coefficient (say, $\alpha_3 <$ 0.4) have no significant effect on the optimal design.
(2) When $\alpha_3 > 0.4$, an increase in α_3 leads to slight increases in both the sample size and the sampling interval, and to a wider control limit.
(3) An increase in α_4 results in an increase in the sample size and a wider control limit.
(4) The optimal sampling interval can be robust to the value of α_4.

Table 4 shows the effect of statistical constraints on the statistical minimum-loss design of the $\bar{x}$ chart. As α_u decreases, the sample size increases, and the control limits become wider. The sampling interval is also affected by the value of α_u, but no specific tendency can be seen. If the lower bound of the power (denoted by p_l) is less than 0.89, then the optimal design remains unchanged. As p_l increases from 0.89, both the sample size and the sampling interval increase.

VI. Conclusions

In this paper, the Burr distribution has been employed to study the statistical minimum-loss design of $\bar{x}$ charts for non-normal data. The Alexander's loss model has been used as the objective function, and the cumulative function of the Burr distribution has been applied to derive the statistical constraints of the problem. An example has been presented to illustrate the solution procedure. From the results of the sensitivity analyses of this example, some important conclusions can be drawn as follows:

(1) Increasing the fixed cost of sampling increases the sampling interval.
(2) Large values of the variable cost of sampling and the cost of finding an assignable cause imply relatively infrequent sampling.
(3) The cost of investigating a false alarm is relatively robust to the optimal design.
(4) Large values of the cost of reworking an out-of-specification part (A) lead to more frequent sam-

Table 3. Effects of Non-normality on the Optimal Design of the $\bar{x}$ Chart

c	q	α_3	α_4	n	h	k	α risk	Power	Cost	Note
6	11	-0.254	3.027	18	1.12	2.84	0.00499	0.91208	88.68704	
6	6	-0.147	3.065	18	1.13	2.85	0.00498	0.91328	88.68079	Group I
5	6	-0.013	3.010	17	1.1	2.8	0.00499	0.90443	88.59777	α_3 : from – to +
5	5	0.040	3.070	17	1.09	2.83	0.00490	0.90086	88.61281	α_4 : close to normal
4	7	0.136	2.979	17	1.1	2.77	0.00487	0.91186	88.55845	
3	11	0.329	3.006	17	1.09	2.84	0.00497	0.90361	88.60141	
4	11	0.050	2.866	17	1.11	2.71	0.00490	0.91873	88.52643	
5	6	-0.013	3.010	17	1.1	2.8	0.00499	0.90443	88.59777	Group II
5	5	0.040	3.070	17	1.09	2.83	0.00490	0.90086	88.61281	α_3 : close to normal
6	4	-0.019	3.169	18	1.12	2.89	0.00494	0.91060	88.69274	α_4 : increasing
7	3	0.005	3.329	18	1.11	2.96	0.00499	0.90136	88.73965	
10	2	0.044	3.646	19	1.14	3.09	0.00498	0.90267	88.85777	
10	10	-0.519	3.462	20	1.16	3.09	0.00499	0.90810	88.95333	
10	7	-0.465	3.430	20	1.17	3.07	0.00491	0.91166	88.93355	Group III
10	3	-0.208	3.418	19	1.14	3.03	0.00497	0.90659	88.83784	α_3 : from – to +
5	3	0.277	3.485	18	1.12	2.98	0.00496	0.90209	88.73532	α_4 : near a constant
3	6	0.484	3.380	19	1.15	3.03	0.00500	0.918860	88.77787	
6	2	0.434	4.106	20	1.17	3.17	0.00493	0.91469	88.91892	Group IV
5	2	0.635	4.630	21	1.19	3.31	0.00495	0.91439	89.04111	α_3 : increasing
2	10	0.884	4.122	22	1.24	3.37	0.00494	0.94192	89.02078	α_4 : near a constant,
2	7	1.014	4.707	23	1.26	3.5	0.00500	0.94315	89.13140	and > 4.0
2	8	0.958	4.443	23	1.27	3.45	0.00494	0.95177	89.08715	
2	7	1.014	4.707	23	1.26	3.5	0.00500	0.94315	89.13140	Group V
2	6	1.094	5.118	24	1.29	3.58	0.00498	0.95280	89.19624	α_3 : close to one
4	2	0.956	5.937	23	1.24	3.55	0.00495	0.91862	89.25489	α_4 : increasing
9	1	1.060	7.215	24	1.26	3.7	0.00500	0.91519	89.38888	
9	1	1.060	7.215	24	1.26	3.7	0.00500	0.91519	89.38888	
2	4	1.432	7.356	27	1.38	3.86	0.00496	0.97176	89.43012	Group VI
2	3	1.909	12.460	29	1.42	4.13	0.00497	0.97170	89.64276	α_3 : increasing, >>0
5	1	2.485	29.560	30	1.41	4.3	0.00497	0.94258	89.90000	α_4 : increasing, >>0
1	9	2.940	19.760	32	1.5	4.78	0.00498	0.99357	89.83807	
1	6	3.810	38.670	34	1.55	5.01	0.00489	1.00000	90.00147	

pling.

(5) A larger mean shift leads to a smaller sample size and a short sampling interval.

(6) Large values of the time required to find an assignable cause correspond to a short sampling interval.

(7) A larger value of the production rate leads to a short sampling interval.

(8) Looser tolerances result in infrequent sampling.

(9) Small values of the skewness coefficient (say, $\alpha_3 < 0.4$) have no significant effect on the optimal design.

(10) When $\alpha_3 > 0.4$, an increase in α_3 leads to slight increases in both the sample size and the sampling interval, and to a wider control limit.

(11) An increase in α_4 results in an increase in the sample size and a wider control limit.

(12) The optimal sampling interval can be robust to the value of α_4.

(13) As the upper bound of the α risk decreases, the sample size may increase, and the control limits may become wider.

(14) As the lower bound of the power increases, both the sample size and the sampling interval may increase.

Table 4. Effects of Statistical Constraints on the Optimal Design of the $\bar{x}$ Chart

		n	h	k	α risk	Power	Cost
	0.0005	30	1.34	4.23	0.00050	0.90214	90.02889
	0.001	27	1.3	3.87	0.00100	0.91833	89.62222
α_u	0.005	19	1.15	3.03	0.00500	0.91886	88.77787
	0.01	16	1.12	2.67	0.00984	0.91907	88.52813
	0.2	14	1.13	2.38	0.01693	0.92505	88.43328
	0.99	25	1.35	3.03	0.00500	0.99149	89.11697
	0.95	21	1.23	3.03	0.00500	0.95537	88.83976
p_l	0.90	19	1.15	3.03	0.00500	0.91886	88.77787
	0.85	18	1.11	3.03	0.00500	0.89451	88.77409
	0.80	18	1.11	3.03	0.00500	0.89451	88.77409

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非常態性資料下平均數管制圖之統計性最小損失設計

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摘　要

當使用$\bar{x}$管制圖以監控製程之品質特性時，必須先決定三個參數，這三個管制圖參數分別爲：樣本數(n)、抽樣之間隔時間(h)和管制界線之係數(k)。在1956年，A.J. Duncan提出$\bar{x}$管制圖之經濟性設計，以決定$\bar{x}$管制圖的三個參數，其目標函數爲使與抽樣與檢驗之相關成本極小化。S.M. Alexander與同事於其1995年的論文中結合Duncan之成本模式和田口之損失函數，提出$\bar{x}$管制圖之最小損失設計。除了經濟性設計之外，另一種設計管制圖的方式稱爲管制圖之統計性設計。在管制圖之統計性設計方面，主要是控制管制圖犯型一錯誤的機率和管制圖的檢定能力。傳統上，當設計管制圖時，均假設量測值呈常態分配。然而，這個假設未必爲眞。在本文中，我們以Burr分配來代表各種不同的非常態分配，並發展在非常態分配下$\bar{x}$管制圖之統計性最小損失設計。由本文之研究結果得知：當偏態係數大於0.4時，偏態係數之遞增會導致較大的樣本數、較長的抽樣間隔和較寬的管制界線；而峰態係數之遞增則會產生較大的樣本數和較寬的管制界線。