THE BUCKLING OF THIN ORTHOTROPIC PLATES UNDER EDGE COMPRESSION

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SUMMARY

This project deals, on theoretical basis, with the buckling of a thin plate of orthotropic material under edge compression in two perpendicular directions. It is solved here for three different sets of boundary conditions. The equation of buckling of the plate is obtained by solving the general differential equation pertaining to the plate. From the condition that the constants of integration in this equation must not all vanish simultaneously a functional relation free from any constants of integration can always be found, which contains only the value of the critical load, the dimension of the plate and the elastic constants of the material. However, an explicit solution of the critical load from the functional relation is generally not able to be deduced and numerical solution has to be used.

The results are illustrated by calculating the coefficient of critical stress for a plate of spruce. The values are plotted and discussed.

I. INTRODUCTION

The buckling of isotropic plates under edge thrust has been treated exhaustively for various loading cases and boundary conditions. However, the anisotropic plates have not been dealt with thoroughly in spite of their frequent occurence in practice as stiffened or corrugated metal or as plywood. The values of buckling stress for an orthotropic plate, loaded by compressive stresses in one direction, have been computed by C. B. Ling and H. S. Tan [1] and R. C. T. Smith [2]; and E. Seydel [3] has investigated of a simply supported orthotropic plate under shear.

In this project we consider a rectangular thin orthotropic plate of thickness h subjected to an edge compression along its two principal directions as shown in Fig. 1. The four edges are assumed to be in three different sets of boundary conditions, i. e.:

1. All edges are simply supported,
2. Three edges simply supported and one edge clamped,
3. Two edges simply supported and two clamped.

When any one or both of the loads P_1 and P_2 are gradually increasing, it will finally arrive at the critical condition that the flat form of equilibrium of the plate becomes unstable and the elastic instability occurs. The loads corresponding to this

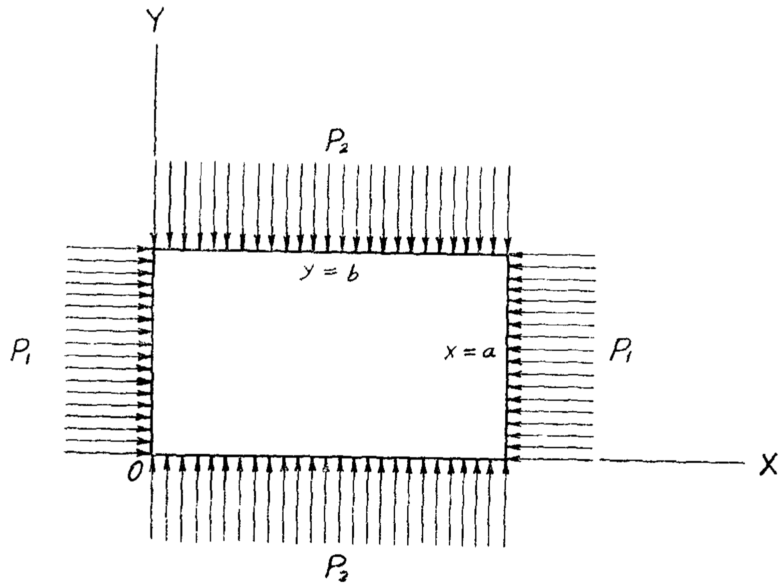

Fig. 1

condition are called as critical loads or buckling loads of the plate.

By assuming the plate buckles or deflects slightly under the action of critical loads, an equation of the plate in this condition can be found from the differential equation of the plate and its boundary conditions. This equation must contain certain coefficients or constants of integration so as to conform to the buckled shape. Then from the condition that these coefficients must not all vanish the magnitude of the required critical loads can be determined, for otherwise the equation would give a flat form of equilibrium.

II. GENERALIZED HOOKE'S LAW

The generalized Hooke's law states that each of the six components of stress at any point of a medium is a linear function of the six components of strain at the point, and conversely [4]. Expressed mathematically, we have the six strain-stress equations of the type

$$\varepsilon_x = c_{11}\sigma_x + c_{12}\sigma_y + c_{13}\sigma_z + c_{14}\tau_{xy} + c_{15}\tau_{yz} + c_{16}\tau_{zx} \tag{1}$$

where $\varepsilon_x, \ldots, \gamma_{xy}, \ldots$ denote six components of strain, $\sigma_x, \ldots, \tau_{xy}, \ldots$ denote six components of stress and $c_{11}, \ldots$ denote 36 elastic constants of the material. If we denote Γ as the strain column vector, T the stress column vector, and C the coefficient matrix, then Eq. (1) can be written in a matrix equation as

$$\Gamma = CT \tag{2}$$

in which

$$\Gamma=\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{zx}\end{bmatrix},\quad T=\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{bmatrix},\quad C=\begin{bmatrix}c_{11} & c_{12}\cdots c_{16}\\ c_{21} & c_{22}\cdots c_{26}\\ \cdots & \cdots\cdots\\ c_{61} & c_{62}\cdots c_{66}\end{bmatrix} \tag{3}$$

The relations which ensure the existence of the strain energy function are:

$$c_{rs}=c_{sr} \quad (r,\ s=1,2,\ \ldots,\ 6) \tag{4}$$

and the number of elastic constants is then reduced from 36 to 21.

For an orthotropic material which possesses at each point three planes of elastic symmetry at right angles to each orther, in taking these planes parallel to the co-ordinate planes, the following 12 coefficients must vanish.

$$c_{14}=c_{15}=c_{16}=c_{24}=c_{25}=c_{26}=c_{34}=c_{35}=c_{36}=c_{45}=c_{46}=c_{56}=0 \tag{5}$$

Thus the number of elastic constants for an orthotropic material reduces to nine only and the Hooke's law reduces to

$$\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{zx}\end{bmatrix}=\begin{bmatrix}c_{11} & c_{12} & c_{13} & 0 & 0 & 0\\ c_{21} & c_{22} & c_{23} & 0 & 0 & 0\\ c_{31} & c_{32} & c_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & c_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & c_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & c_{66}\end{bmatrix}\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{bmatrix} \tag{6}$$

in which

$$c_{rs}=c_{sr} \quad (r,\ s=1,\ 2,\ 3) \tag{7}$$

The moduli along principal directions, i.e. direction perpendicular to planes of symmetry, can be obtained as follows:

If we suppose that all the stress components vanish except σ_x, we have

$$\varepsilon_x=c_{11}\sigma_x \tag{8}$$

so that the modulus of elasticity corresponding to this direction is

$$E_x=\frac{1}{c_{11}} \quad \text{or} \quad c_{11}=\frac{1}{E_x} \tag{9}$$

Similarly

$$E_y=\frac{1}{c_{22}},\quad c_{22}=\frac{1}{E_y}$$
$$E_z=\frac{1}{c_{33}},\quad c_{33}=\frac{1}{E_z} \tag{10}$$

In the same case the Poisson's ratio of the contraction in the direction of y to the extension in the direction of x is

$$\nu_{xy}=-\frac{c_{12}}{c_{11}} \quad \text{or} \quad c_{12}=-\frac{\nu_{xy}}{E_x} \tag{11}$$

Similary

$$
\begin{aligned}
\nu_{xz} &= -\frac{c_{13}}{c_{11}}, \quad c_{13} = -\frac{\nu_{xy}}{E_x} \\
\nu_{zx} &= -\frac{c_{31}}{c_{33}}, \quad c_{31} = -\frac{\nu_{zx}}{E_z} \\
\nu_{zy} &= -\frac{c_{32}}{c_{33}}, \quad c_{32} = -\frac{\nu_{zy}}{E_z} \\
\nu_{yz} &= -\frac{c_{23}}{c_{22}}, \quad c_{23} = -\frac{\nu_{yz}}{E_y} \\
\nu_{yx} &= -\frac{c_{21}}{c_{22}}, \quad c_{21} = -\frac{\nu_{yx}}{E_y}
\end{aligned}
\tag{12}
$$

and using the relations involved in Eq. (7), we obtain the following identites

$$
\begin{aligned}
\frac{\nu_{xy}}{E_x} &= \frac{\nu_{yx}}{E_y}, \\
\frac{\nu_{yz}}{E_y} &= \frac{\nu_{zy}}{E_z}, \\
\frac{\nu_{zx}}{E_z} &= \frac{\nu_{xz}}{E_x}
\end{aligned}
\tag{13}
$$

If on the other hand we suppose that all the stress components vanish except τ_{xy}, we have

$$\gamma_{xy} = c_{44}\tau_{xy} \tag{14}$$

so that the modulus of rigidity corresponding to the pair of directions x, y is

$$G_{xy} = \frac{1}{c_{44}} \quad \text{or} \quad c_{44} = \frac{1}{G_{xy}} \tag{15}$$

Similarly

$$
\begin{aligned}
G_{yz} &= \frac{1}{c_{55}}, \quad c_{55} = \frac{1}{G_{yz}} \\
G_{zx} &= \frac{1}{c_{66}}, \quad c_{66} = \frac{1}{G_{zx}}
\end{aligned}
\tag{16}
$$

From the results shown in Eqs. (9), (10), (11), (12), (15) and (16) the Hooke's law for orthotropic material Eq. (6) can be rewritten as follows:

$$
\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}
=
\begin{bmatrix}
\frac{1}{E_x} & -\frac{\nu_{xy}}{E_x} & -\frac{\nu_{xz}}{E_x} & 0 & 0 & 0 \\
-\frac{\nu_{yx}}{E_y} & \frac{1}{E_y} & -\frac{\nu_{yz}}{E_y} & 0 & 0 & 0 \\
-\frac{\nu_{zx}}{E_z} & -\frac{\nu_{zy}}{E_z} & \frac{1}{E_z} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{G_{xy}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G_{yz}} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{G_{zx}}
\end{bmatrix}
\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix}
\tag{17}
$$

III. MODULI ALONG ANY DIRECTION

Let x, y, z and x', y', z' be the coordinates of a point referred to two different systems of rectangular axes, of which the direction cosines are connected to one another by the left table.

	x	y	z
x′	l_1	m_1	n_1
y′	l_2	m_2	n_2
z′	l_3	m_3	n_3

It can be shown that the moduli of an anisotropic material in general corresponding to the second system of axes can be expressed in terms of those corresponding to the first system of axes in the following forms [4].

For modulus of rigidity $G_{y'z'}$, corresponding to directions y', z'.

$$\frac{1}{G_{y'z'}}=[c_{11},\ c_{22},\ldots,\ c_{12},\ldots]\ (2l_2l_3,\ 2m_2m_3,\ 2n_2n_3,\ l_2m_3+l_3m_2,\ m_2n_3+m_3n_2,\ n_2l_3+n_3l_2)^2 \tag{18}$$

The right hand side of the equation represents a complete quadratic function of the six arguments $2l_2l_3,\ldots$ with coefficients c_{11}, $c_{22},\ldots$ consisting of altogether twenty one terms.

For modulus of elasticity $E_{x'}$ corresponding to direction x'.

$$\frac{1}{E_{x'}}=[c_{11},\ c_{22},\ldots,\ c_{12},\ldots]\ \left(l_1^2,\ m_1^2,\ n_1^2,\ l_1m_1,\ n_1n_1,\ n_1l_1\right)^2 \tag{19}$$

Similarly, the right hand side of the equation represents a complete quadratic function of the six arguments $l_1^2,\ldots$ with coefficients $c_{11},\ldots$

For Poisson's ratio corresponding to contraction in direction i' $(l,\ m,\ n)$ at right angle to x':

$$\nu_{x'i'}=-\frac{1}{2\phi}\left[l^2\frac{\partial\phi}{\partial(l_1^2)}+m^2\frac{\partial\phi}{\partial(m_1^2)}+n^2\frac{\partial\phi}{\partial(n_1^2)}+lm\frac{\partial\phi}{\partial(l_1m_1)}+mn\frac{\partial\phi}{\partial(m_1n_1)}+nl\frac{\partial\phi}{\partial(n_1l_1)}\right] \tag{20}$$

where ϕ is the quadratic function in Eq. (19) of arguments $l_1^2,\ldots$, and the differential coefficients are formed as if these arguments were independent.

In the case of orthotropic material, if we take the first system of axes $(x,\ y,\ z)$ lie in the principal directions of the material it readily follows that the expressions for the moduli reduce to as follows:

$$\frac{1}{G_{y'z'}}=4\left[\frac{l_2^2l_3^2}{E_x}+\frac{m_3^2m_3^2}{E_y}+\frac{n_2^2n_3^2}{E_z}-2l_2l_3m_2m_3\frac{\nu_{xy}}{E_x}-2m_2m_3n_2n_3\frac{\nu_{yz}}{E_y}-2n_2n_3l_2l_3\frac{\nu_{zx}}{E_z}+\frac{(l_2m_3+l_3m_2)^2}{G_{xy}}+\frac{(m_2n_3+m_3n_2)^2}{G_{yz}}+\frac{(n_2l_3+n_3l_2)^2}{G_{zx}}\right] \tag{21}$$

$$\frac{1}{E_{x'}}=\frac{l_1^4}{E_x}+\frac{m_1^4}{E_y}+\frac{n_1^4}{E_z}-2l_1^2m_1^2\frac{\nu_{xy}}{E_x}-2m_1^2n_1^2\frac{\nu_{yz}}{E_y}-2n_1^2l_1^2\frac{\nu_{zx}}{E_z}+\frac{l_1^2m_1^2}{G_{xy}}+\frac{m_1^2n_1^2}{G_{yz}}+\frac{n_1^2l_1^2}{G_{zx}} \tag{22}$$

$$\nu_{x'i'} = -E_{x'}\Big[\frac{l^2 l_1^2}{E_x} + \frac{m^2 m_1^2}{E_y} + \frac{n^2 n_1^2}{E_z} - \left(l^2 m_1^2 + l_1^2 m^2\right)\frac{\nu_{xy}}{E_x} + \left(m^2 n_1^2 + m_1^2 n^2\right)\frac{\nu_{yz}}{E_y}$$
$$-\left(m^2 n_1^2 + m_1^2 n^2\right)\frac{\nu_{yz}}{E_y} - \left(n^2 l_1^2 + n_1^2 l^2\right)\frac{\nu_{zx}}{E_z} + \frac{l l_1 m m_1}{G_{xy}} + \frac{m m_1 n n_1}{G_{yz}} + \frac{n n_1 l l_1}{G_{zx}}\Big] \quad (23)$$

IV. DIFFERENTIAL EQUATION OF THE PLATE

Referring to Fig. 2, let T, S, N denote the components of stress-resultant, and H, K the components of stress-conples, acting on unit length of the plate edge. To neglect the body forces and for no surface traction, it can be readily shown from statics that the following sets of equations of equilibrium exist by equating to zero the sum of forces and moment components along three axes respectively. For force components, we have

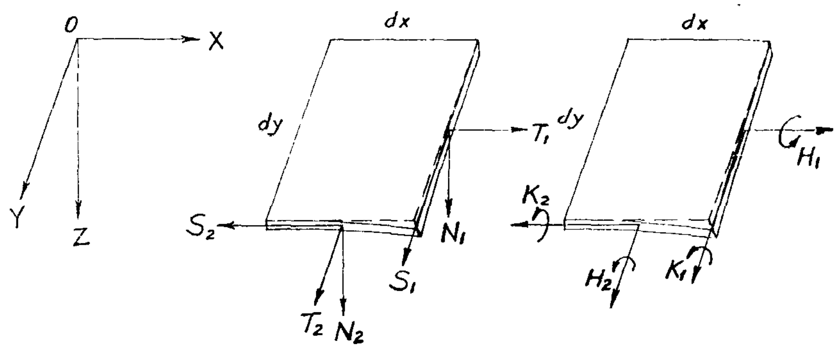

Fig. 2

$$\begin{aligned}
&\frac{\partial T_1}{\partial x} - \frac{\partial S_2}{\partial y} - N_1\frac{\partial^2 w}{\partial x^2} - N_2\frac{\partial^2 w}{\partial y \partial x} = 0\\
&\frac{\partial S_1}{\partial x} + \frac{\partial T_2}{\partial y} - N_1\frac{\partial^2 w}{\partial x \partial y} - N_2\frac{\partial^2 w}{\partial y^2} = 0 \qquad (24)\\
&\frac{\partial N_1}{\partial x} + \frac{\partial N_2}{\partial y} + T_1\frac{\partial^2 w}{\partial x^2} - S_2\frac{\partial^2 w}{\partial y \partial x} + S_1\frac{\partial^2 w}{\partial x \partial y} + T_1\frac{\partial^2 w}{\partial y^2} = 0
\end{aligned}$$

And for moment components,

$$\begin{aligned}
&\frac{\partial H_1}{\partial x} - \frac{\partial K_2}{\partial y} + N_2 = 0\\
&\frac{\partial K_1}{\partial x} + \frac{\partial H_2}{\partial y} - N_1 = 0 \qquad (25)\\
&K_1\frac{\partial^2 w}{\partial x \partial y} - K_2\frac{\partial^2 w}{\partial y \partial x} + H_1\frac{\partial^2 w}{\partial x^2} + H_2\frac{\partial^2 w}{\partial y^2} + S_1 + S_2 = 0
\end{aligned}$$

It is noted that corresponding to the deflection w at any point (x, y) on the plane, the quantities

$$\frac{\partial^2 w}{\partial x^2}\,dx \;,\quad \frac{\partial^2 w}{\partial y^2}\,dy$$

denote the change of slope along dx, dy; and

$$\frac{\partial^2 w}{\partial y \partial x}\,dy \;,\quad \frac{\partial^2 w}{\partial x \partial y}\,dx$$

denote the angle of rotation of dx, dy about the axes of y and x respectively.

The differential equation satisfied by the deflection w of the plate may be obtained from Eqs. (24) and (25) by putting the stress-resultants T, S, N and the stresscouples H, K in terms of the deflection w.

As shown in Fig. 3, we have the displacement:

$$u=-z\frac{\partial w}{\partial x} \tag{26}$$

In the same manner

$$v=-z\frac{\partial w}{\partial y} \tag{27}$$

The components of strain are:

$$\varepsilon_x=\frac{\partial u}{\partial x}=-z\frac{\partial^2 w}{\partial x^2}$$

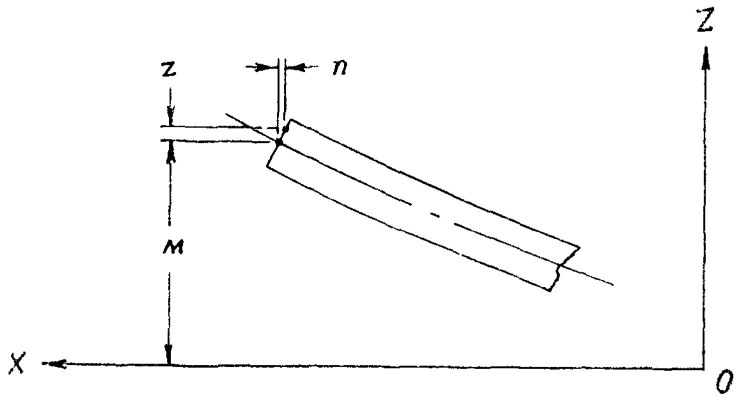

Fig. 3

$$\begin{aligned}\varepsilon_y&=\frac{\partial v}{\partial y}=-z\frac{\partial^2 w}{\partial y^2}\ ,\\ \gamma_{xy}&=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=-2z\frac{\partial^2 w}{\partial x\partial y}\end{aligned} \tag{28}$$

But from the generalized Hooke's law

$$\begin{aligned}\varepsilon_x&=\frac{\sigma_x}{E_x}-\nu_{yx}\frac{\sigma_y}{E_y}\ ,\\ \varepsilon_y&=\frac{\sigma_y}{E_y}-\nu_{xy}\frac{\sigma_x}{E_x}\ ,\\ \gamma_{xy}&=\frac{\tau_{xy}}{G_{xy}}\end{aligned} \tag{29}$$

By substituting Eqs. (28) into Eqs. (29) we thus obtain

$$\begin{aligned}\sigma_x&=-\frac{zE_x}{1-\nu_{xy}\,\nu_{yx}}\left(\frac{\partial^2 w}{\partial x^2}+\nu_{yx}\frac{\partial^2 w}{\partial y^2}\right)\ ,\\ \sigma_y&=-\frac{zE_y}{1-\nu_{xy}\,\nu_{yx}}\left(\frac{\partial^2 w}{\partial y^2}+\nu_{xy}\frac{\partial^2 w}{\partial x^2}\right)\ ,\\ \tau_{xy}&=-2z\,G_{xy}\frac{\partial^2 w}{\partial x\partial y}\end{aligned} \tag{30}$$

from which

$$K_1=\int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma_x dz=-D_1\left(\frac{\partial^2 w}{\partial x^2}+\nu_{yx}\frac{\partial^2 w}{\partial y^2}\right) \tag{31}$$

where

$$D_1=\frac{E_x h^3}{12(1-\nu_{xy}\ \nu_{yx})} \tag{32}$$

Similarly

$$K_2=-D_2\left(\frac{\partial^2 w}{\partial y^2}+\nu_{xy}\frac{\partial^2 w}{\partial x^2}\right) \tag{33}$$

where

$$D_2=\frac{E_y h^3}{12(1-\nu_{xy}\ \nu_{yx})} \tag{34}$$

And

$$H_1=-H_2=D_3\frac{\partial^2 w}{\partial x\partial y} \tag{35}$$

where

$$D_3=\frac{h^3}{6}G_{xy} \tag{36}$$

By neglecting the higher order terms and by making use of Eqs. (24) and (25), we obtain the deflection w of a thin orthotropic plate over the region R satisfics the folloing equation

$$D_1\frac{\partial^4 w}{\partial x^4}+2K\frac{\partial^4 w}{\partial x^2\partial y^2}+D_2\frac{\partial^4 w}{\partial y^4}=T_1\frac{\partial^2 w}{\partial x^2}+2S_1\frac{\partial^2 w}{\partial x\partial y}+T_2\frac{\partial^2 w}{\partial y^2} \tag{37}$$

in which

$$K=\frac{1}{2}(\nu_{xy}D_1+\nu_{xy}D_2)+D_3 \tag{38}$$

In the case to be considered, we have

$$\begin{aligned} S_1&=S_2=0\\ T_1&=-P_1\\ T_2&=-P_2 \end{aligned} \tag{39}$$

where P_1 and P_2 represent the compressive loads per unit length. The differential equation (37) then becomes

$$D_1\frac{\partial^4 w}{\partial x^4}+2K\frac{\partial^4 w}{\partial x^2\partial y^2}+D_2\frac{\partial^4 w}{\partial y^4}+P_1\frac{\partial^2 w}{\partial x_2}+P_2\frac{\partial^2 w}{\partial y^2}=0 \tag{40}$$

V. BOUNDARY CONDITIONS

The boundary conditions of $x=0$ and $x=a$ are assumed to be simply supported. Whereas $y=0$ and $y=b$ are considered as simply supported or clamped. The three cases to be considered in this project are:

1. All edges simply supported;
2. One edge clamped, three edges simply supported;

3. Two opposite edges clamped, two opposite edges simply supported.

Along simply supported edges both the deflection and flexural couple are zero, i. e.

$$w=0 \tag{41}$$

and

$$K_1=K_2=0 \tag{42}$$

or by Eqs. (31) and (33), we have

$$\begin{aligned} &\frac{\partial^2 w}{\partial x^2}+\nu_{yx}\frac{\partial^2 w}{\partial y^2}=0,\\ &\frac{\partial^2 w}{\partial y^2}+\nu_{xy}\frac{\partial^2 w}{\partial x^2}=0 \end{aligned} \tag{43}$$

For clamped edges it is obvious,

$$w=0 \tag{44}$$

and

$$\frac{\partial w}{\partial y}=0 \tag{45}$$

VI. GENERAL SOLUTION

The general solution of Eq. (40) may be assumed in the form:

$$w=F(y)\ \sin\frac{m\pi x}{a} \tag{46}$$

in which $F(y)$ is a function of y only and m is an positive integer. This clearly satisfies the boundary conditions for $x=0$ and $x=a$, it gives

$$w=0,$$

$$\frac{\partial^2 w}{\partial x^2}+\nu_{yx}\frac{\partial^2 w}{\partial y^2}=0$$

Then by substituting the assumed deflection (46) into Eq. (40), we can readily obtain a fourth order ordinary differential equation for the function of $F(y)$ such as

$$\frac{d^4F}{dy^4}+\frac{2}{D_2}\left(\frac{P_2}{2}-\frac{m^2\pi^2K}{a^2}\right)\frac{d^2F}{dy^2}+\frac{m^2\pi^2}{a^2D_2}\left(\frac{m^2\pi^2D_1}{a^2}-P_1\right)=0 \tag{47}$$

From which, the general solution can easily be found by the most familiar method. Thus

$$F(y)=c_1e^{-\alpha y}+c_2e^{\alpha y}+c_3\ \cos\beta y+c_4\ \sin\beta y \tag{48}$$

with

$$\begin{aligned} \alpha&=\frac{1}{\sqrt{D_2}}\left\{\frac{m^2\pi^2K}{a^2}-\frac{P_2}{2}+\left[\left(\frac{P_2}{2}-\frac{m^2\pi^2K}{a^2}\right)^2-\frac{m^2\pi^2D_2}{a^2}\left(\frac{m^2\pi^2D_1}{a^2}-P_1\right)\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}},\\ \beta&=\frac{1}{\sqrt{D_2}}\left\{\frac{P_2}{2}-\frac{m^2\pi^2K}{a^2}+\left[\left(\frac{P_2}{2}-\frac{m^2\pi^2K}{a^2}\right)^2-\frac{m^2\pi^2D_2}{a^2}\left(\frac{m^2\pi^2D_1}{a^2}-P_1\right)\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}} \end{aligned} \tag{49}$$

The constants of integration c_1, c_2, c_3, and c_4 must be determined in each particular case from the boundary conditions along the edges $y=0$ and $y=b$.

From each set of boundary conditions, a functional relation called as the characteristic equation always exists between α and β, such that

$$f(\alpha b,\ \beta b)=0 \tag{50}$$

Since both α and β contain P_1 and P_2, these characteristic equations can be used to calculate the critical values of P's provided the dimension of the plate and the elastic constants of the material are known.

VII. DETERMINATION OF CHARACTERISTIC EQUATIONS

1. All edges simply supported

When the edges along $y=0$ and $y=b$ are simply supported, the boundary conditions are

$$w=0,\ \frac{\partial^2 w}{\partial y^2}+\nu_{xy}\frac{\partial^2 w}{\partial x^2}=0$$

The boundary conditions along $y=0$ are satisfied if we take in the general solution (48)

$$c_1=-c_2,\ c_3=0 \tag{51}$$

By replacing

$$2c_2=\mathrm{A},\ c_4=\mathrm{B} \tag{52}$$

the function $\mathrm{F}(y)$ given in (48) can then be written in the form

$$\mathrm{F}(y)=\mathrm{A}\ \sinh\ \alpha y+\mathrm{B}\ \sin\beta y \tag{53}$$

i. e.

$$w=(\mathrm{A}\ \sinh\ \alpha y+\mathrm{B}\ \sin\ \beta y)\ \sin\ \frac{m\pi x}{a} \tag{54}$$

For $y=b$ it follows that

$$\mathrm{A}\ \sinh\ \alpha b+\mathrm{B}\ \sin\ \beta b=0, \tag{55}$$

$$\mathrm{A}\left(\alpha^2-\nu_{xy}\frac{m^2\pi^2}{a^2}\right)\sinh\ \alpha b-\mathrm{B}\left(\beta^2+\nu_{xy}\frac{m^2\pi^2}{a^2}\right)\sin\ \beta b=0$$

The condition that A and B not simultaneously equal to zero should the determinant of these equations becomes zero, thus

$$\begin{vmatrix} \sinh\ \alpha b & \sin\ \beta b \\ \left(\alpha^2-\nu_{xy}\frac{m^2\pi^2}{a^2}\right)\sinh\ \alpha b & -\left(\beta^2+\nu_{xy}\frac{m^2\pi^2}{a^2}\right)\sin\ \beta b \end{vmatrix}=0 \tag{56}$$

On reducing we find

$$(\alpha^2+\beta^2)\ \sinh\ \alpha b\ \sin\ \beta b=0 \tag{57}$$

Since neither the factor $\alpha^2+\beta^2$ nor $\sinh\ \alpha b$ vanishes, we must have

$$\sin\ \beta b=0 \tag{58}$$

or

$$\beta=\frac{n\pi}{b} \tag{59}$$

in which n is an integer.

By substituting back Eq. (59) into Eq. (55), we have

$$\mathrm{A}=0 \tag{60}$$

So that

$$F(y)=B\sin\frac{n\pi y}{b} \tag{61}$$

and

$$w=b\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{62}$$

For the determination of critical loads we substituting the value of β in Eq. (59) into the second equation of Eqs. (49) and obtain

$$\frac{n^2\pi^2}{b^2}=\frac{1}{D_2}\left\{\frac{P_{2cr}}{2}-\frac{m^2\pi^2K}{a^2}+\left[\left(\frac{P_{2cr}}{2}-\frac{m^2\pi^2K}{a^2}\right)-\frac{m^2\pi^2D_2}{a^2}\left(\frac{m^2\pi^2D_1}{a^2}-P_{1cr}\right)\right]^{\frac{1}{2}}\right\} \tag{63}$$

Hence

$$P_{1cr}+\left(\frac{na}{mb}\right)^2P_{2cr}=\frac{n^2\pi^2D_1}{b^2}\left[\left(\frac{mb}{na}\right)^2+\frac{2K}{D_1}+\frac{D_2}{D_1}\left(\frac{na}{mb}\right)^2\right] \tag{64}$$

which is the simplified equation of critical loads. It may be treated in the following two methods:

(a) With a given a/b ratio, we can investigate the relation between P_{1cr} and P_{2cr} due to the various values of m and n, and for the case of a square plate (a/b =1) then Eq. (64) reduces to

$$P_{1cr}+\left(\frac{n}{m}\right)^2P_{2cr}=\frac{n^2\pi^2D_1}{b^2}\left[\left(\frac{m}{n}\right)^2+\frac{2K}{D_1}+\frac{D_2}{D_1}\left(\frac{n}{m}\right)^2\right] \tag{65}$$

Since

$$P_{cr}=p_{cr}\,h \tag{66}$$

where p_{cr} denotes the critical load per unit length and h the thickness of the plate, or

$$p_{1cr}+\left(\frac{n}{m}\right)^2p_{2cr}=\frac{n^2\pi^2D_1}{b^2h}\left[\left(\frac{m}{n}\right)^2+\frac{2K}{D_1}+\frac{D_2}{D_1}\left(\frac{n}{m}\right)^2\right] \tag{67}$$

(b) If $P_{2cr}/P_{1cr}=\gamma$ is definite, then Eq. (64) may be deduced in the form

$$p_{1cr}=\frac{n^2\pi^2D_1}{b^2h}\left[\left(\frac{mb}{na}\right)^2+\frac{2K}{D_1}+\frac{D_2}{D_1}\left(\frac{na}{mb}\right)^2\right]\left[1+\gamma\left(\frac{na}{mb}\right)^2\right]^{-1} \tag{68}$$

in which the integers m and n should be so adjusted that it gives the critical load as a minimum.

Eq. (68) shows that for a given plate the magnitude of critical loads depends upon the number of half waves in the load directions and the ratio a/b.

It is obvious that the smallest value of p_{1cr} will be obtained by taking $m=n=1$. The physical meaning of this is that a plate buckles in such a form that there can be only one half wave in both perpendicular direction. Thus the expression for the critical stress becomes

$$p_{1cr}=\frac{\pi^2D_1}{b^2h}\left[\left(\frac{b}{a}\right)^2+\frac{2K}{D_1}+\frac{D_2}{D_1}\left(\frac{a}{b}\right)^2\right]\left[1+\gamma\left(\frac{a}{b}\right)^2\right]^{-1} \tag{69}$$

2. One edge clamped and three edges simply supported.

Since the boundary conditions for simply supported edges $x=0$ and $x=a$ are satisfied. For the edges $y=0$ and $y=b$ we assume that $y=0$ is simply supported and

$y=b$ clamped. To satisfy the boundary conditions along $y=0$, we have the same deflection pattern as in Eq. (54), i. e.

$$w=(\mathrm{A}\sinh\alpha y+\mathrm{B}\sin\beta y)\sin\frac{m\pi x}{a}$$

And along $y=b$, the boundary conditions are:

$$w=0,\ \frac{\partial w}{\partial y}=0$$

it shows that

$$\mathrm{A}\sinh\alpha b+\mathrm{B}\sin\beta b=0,$$
$$\mathrm{A}\alpha\cosh\alpha b+\mathrm{B}\beta\cos\beta b=0 \tag{70}$$

By following the same reasoning the buckled form of equilibrium of the plate is possible only when the determinant of these equations becomes zero. Thus

$$\beta\sinh\alpha b\cos\beta b-\alpha\cosh\alpha b\sin\beta b=0 \tag{71}$$

or the required characteristic equation is

$$\tanh\alpha b=\frac{\alpha}{\beta}\tan\beta b \tag{72}$$

It appears in this and the subsequent case that the critical load can not be solved explicitly from the characteristic equation and therefore numerical solution has to be obtained by cut and try method. Rewrite Eq. (72) in the form

$$\frac{\tanh\alpha b}{\alpha b}=\frac{\tan\beta b}{\beta b} \tag{73}$$

then plot curves by taking respectively both the sides of above equation as ordinates and αb and βb as abscissas, we therefore obtain a series of intersecting points. A pair of smallest possible values we chosen for use is

$$\alpha b=\beta b=\frac{5\pi}{4} \tag{74}$$

3. Two opposite edges clamped and two opposite edges simply supported.

Let both the edges $x=0$ and $x=a$ are simply supported, then the deflection curve, w, is as shown in Eq. (46) in which the function $\mathrm{F}(y)$ is given in Eq. (48). If both the edges $y=0$ and $y=b$ are clamped, the boundary conditions of that portion of the plate are

$$w=0,\ \frac{\partial w}{\partial y}=0$$

From the conditions along the edge $y=0$, it readily shows us that

$$c_1=-\frac{\alpha c_3-\beta c_4}{2\alpha}$$

$$c_2=-\frac{\alpha c_3+\beta c_4}{2\alpha} \tag{75}$$

By replacing c_3 and c_4 by A and B respectively, we have from Eqs. (46) and (48).

$$\mathrm{F}(y)=\mathrm{A}(\cos\beta y-\cosh\alpha y)+\mathrm{B}(\sin\beta y-\frac{\beta}{\alpha}\sinh\alpha y),$$

$$w=\left[A(\cos\beta y-\cosh\alpha y)+B(\sin\beta y-\frac{\beta}{\alpha}\sinh\alpha y)\right]\sin\frac{m\pi x}{a} \tag{76}$$

From the boundary conditions for $y=b$,

$$\begin{aligned}&A(\cos\beta b-\cosh\alpha b)+B(\sin\beta b-\frac{\beta}{\alpha}\sinh\alpha b)=0,\\&-A\beta(\sin\beta b+\sinh\alpha b)+B\beta\,(\cos\beta b-\cosh\alpha b)=0\end{aligned} \tag{77}$$

Similarly, the determinant of Eqs. (77) must vanish in order to conform with the buckled form of equilibrium and thus we obtain the characteristic equation in this case as

$$\cosh\alpha b\cos\beta b+\frac{1}{2}\left(\frac{\beta b}{\alpha b}-\frac{\alpha b}{\beta b}\right)\sinh\alpha b\sin\beta b=1 \tag{78}$$

We see that is

$$\beta b=2n\pi \tag{79}$$

then

$$\alpha b=0 \tag{80}$$

in Eq. (79) we choose $n=1$ to determine the minimum buckling stress.

VIII. THE CRITICAL STRESS

Using the forgoing results, we will be able to calculate the critical values of p_1 and p_2.

Except the first case the characteristic equations may be solved by the method of sucessive approximation in which we first assign a value to βb, then the corresponding value of αb can be found by solving the characteristic equation numerically. Hence from the equations of Eq. (49) and the value of the buckling coefficient, k, of the critical stress

$$p_{cr}=k\frac{\pi^2 D_1}{b^2 h} \tag{81}$$

corresponding to each value of a/b can thus be calculated.

It is noted here that in cases 2 and 3 the solutions of p_{1cr} and p_{2cr} can be solved independently. And only the expression of p_{1cr} in case 2 can thus be calculated according to a successive values of m, of which the buckling coefficient, k, may be expressed in the form

$$k=\frac{m^2b^2}{a^2}+C\frac{a^2}{m^2b^2} \tag{82}$$

where C is a combined constant of elastic constants and the solutions of α and β. The transition point for number of waves increasing form m to $m+1$ will be obtained by equation

$$\frac{m^2b^2}{a^2}+C\frac{a^2}{m^2b^2}=\frac{(m+1)^2b^2}{a^2}+C\frac{a^2}{(m+1)^2b^2} \tag{83}$$

from which we get the ratio

$$\frac{a}{b}=C^{-\frac{1}{4}}\left[m(m+1)\right]^{\frac{1}{2}} \tag{84}$$

and

$$C=\frac{625}{256}\cdot\frac{D_2}{D_1} \tag{85}$$

While in other cases the plate buckles in single wave form.

In plotting the curves it is only necessary to plot the curve corresponding to $m=1$, for the curves corresponding to $m=2$, 3, etc. can be obtained from the curve $m=1$ by keeping the ordinates unchanged and doubling, tripling, etc. the abscissas. This statement can easily be verified by Eq. (82).

IX. NUMERICAL EXAMPLE

Refer to C. F. Jenkin's Report on the Material of Construction Used in Aircraft and Aircraft Engines, 1920, we have the following elastic data for spruce:

$E_x=1.95\times10^6$ lbs/sq. in.

$E_y=0.45\times10^6$ lbs/sq. in.

$G_{xy}=0.104\times10^6$ lbs/sq. in.

$\nu_{xy}=0.45$

$\nu_{yx}=0.03$

The values of the buckling coefficient, k, for various sets of boundary conditions will be calculated and plotted. They are tabulated in Table I, of which we assume m or m, $n=1$ and $\gamma=1$.

For case 1,

$$k_{p1}=k_{p2}=\left[1+\left(\frac{a}{b}\right)^2\right]^{-1}\left[\left(\frac{b}{a}\right)^2+\frac{2K}{D_1}+\frac{D_2}{D_1}\left(\frac{a}{b}\right)^2\right] \tag{86}$$

For case 2,

$$\left.\begin{aligned}k_{p1}&=\left(\frac{b}{a}\right)^2+\frac{625}{256}\,\frac{D_2}{D_1}\left(\frac{a}{b}\right)^2,\\ k_{p2}&=\frac{2K}{D_1}\left(\frac{b}{a}\right)^2\end{aligned}\right\} \tag{87}$$

And for case 3,

Table I

	1. All edges simply supported	2. One clamped, three simply supported		3. Two clamped, two simply supported	
$\frac{a}{b}$	$k_{p1,p2}$ $(m=n=\gamma=1)$	k_{p1} $(m=1)$	k_{p2} $(m=1)$	k_{p1} $(m=1)$	k_{p2} $(m=n=1)$
0.2	24.378	25.023	8.608	25.000	9.551
0.4	5.716	6.340	2.152	6.250	3.095

	1. All edges simply supported	2. One clamped, three simply supported		3. Two clamped, two simply supported	
$\frac{a}{b}$	$k_{p_1 \cdot p_2}$ ($m=n=\gamma=1$)	k_{p_1} ($m=1$)	k_{p_2} ($m=1$)	k_{p_1} ($m=1$)	k_{p_2} ($m=n=1$)
0.6	2.357	2.979	0.956	2.778	1.890
0.8	1.253	1.923	0.538	1.563	1.481
1.0	0.788	1.564	0.344	1.000	1.287
1.154*	--	1.501	--	--	--
1.2	0.562	1.506	0.239	0.694	1.182
1.4	0.442	1.615	0.176	0.510	1.119
1.6	0.372	1.833	0.134	0.391	1.077
1.8	0.330	2.134	0.106	0.309	1.049
2.0	0.304	2.504	0.086	0.250	1.029
2.5	0.269	--	0.055	0.160	0.998
3.0	0.253		0.038	0.111	0.981
4.0	0.241	-	0.022	1.063	0.965
8.0	0.233	-	0.005	0.016	0.948
16.0	0.232	--	0.001	0.004	0.944

In which the star notation denotes the minimum point of k.

$$k_{p1}=\left(\frac{b}{a}\right)^2 ,$$
$$k_{p2}=\frac{4D_2}{D_1}+\frac{2K}{D_1}\left(\frac{b}{a}\right)^2 \tag{88}$$

All those values of the buckling coefficients are plotted as shown in Figs. 4, 5 and 6 respectively.

With the aid of Eq. (67), the relation between p_1 and p_2 of a square plate varied as m and n is also plotted as shown in Fig. 7 for the case of all edges which are simply supported.

X. CONCLUSIONS

A. With references to Figs. 4, 5 and 6 the following conclusions may be drawn for the buckling stress p against the value of a/b.

(1) Almost all the curves show that the plate buckles only in a single wave in order that both the perpendicular directions of the plate are compressed.

(2) Except for the case of all edges are simply supported, p_{1cr} and p_{2cr} may be solved independently, so that the curves of buckling coefficients k of p_{1cr} and p_{2cr} for the other two cases are plotted respectively in Figs. 5 and 6.

(3) For the ratio of a/b is smaller than 2, we see that the buckling coefficients k drop very rapidly from infinity. If it is greater than 2 then the curve is decreasing slowly toward to the infiity of a/b.

(4) As the case of two opposite edges clamped and two opposite edges simply

supported, it is shown in Fig. 6, we have $k_{p_{1cr}} = k_{p_{2cr}} = 1.437$ at a/b=0.834.

(5) The foregoing results are generally valid for any kind of orthotropic materials, including also the particular case of isotropic material. Moreover, that these results will also be useful in the investigation of elastic stability of plied plates, such as plywood, ply-bamboo, etc. if properly modified elastic constants are used instead.

B. The relation between p_{1cr} and p_{2cr} for various values of m and n in the case of a square plate are shown in Fig. 7. The values of m and n are indicated on these lines and positive values of p_{1cr} and p_{2cr} indicate compressive stresses. Hence some conclusions are obtained as follows:

(1) Since we seek the smallest values of p_{1cr} and p_{2cr} at which buckling may occur, we need to consider only the portions of the straight lines shown in the figure by full lines and forming the polygon ABCDEF.

(2) By preparing a figure analogous to Fig. 7 for any given ratio a/b, the corresponding critical values of p_{1cr} and p_{2cr} can be obtained from that figure.

(3) When $p_1 = p_2 = p$, we draw through the origin 0 a line which makes an angle of 45° with the horizontal axis. Then the intersection of this line with the line CD determines the critical value of p in this case.

(4) For any value of p_1 the critical value of p_2 is obtained by drawing a vertical line through the corresponding point on the axis of abscissas. The ordinate of the point of intersection of this line with the polygon ABCDEF gives the value of p_{2cr}. If, in the presenting case, p_1 is larger than 1.575 $(\pi^2 D_1/b^2 h)$, p_{2cr} becomes negative. This shows that the plate can stand a compressive stress larger than the critical value for the case of simple compression, provided an adequate tensile stress acts in the perpendicular direction.

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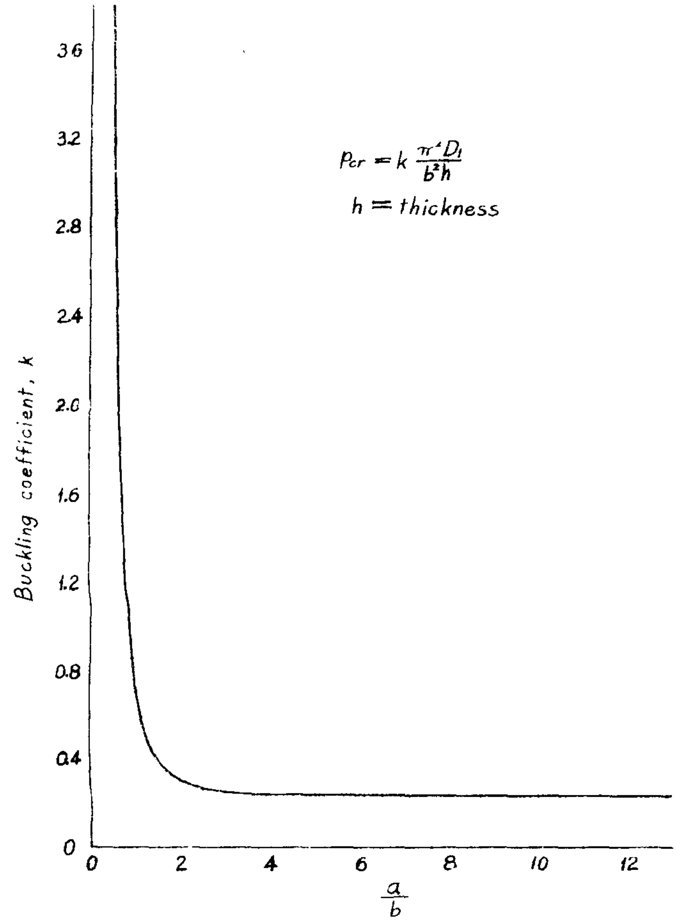
Fig. 4. Buckling coefficient for all edges simply supported plate.



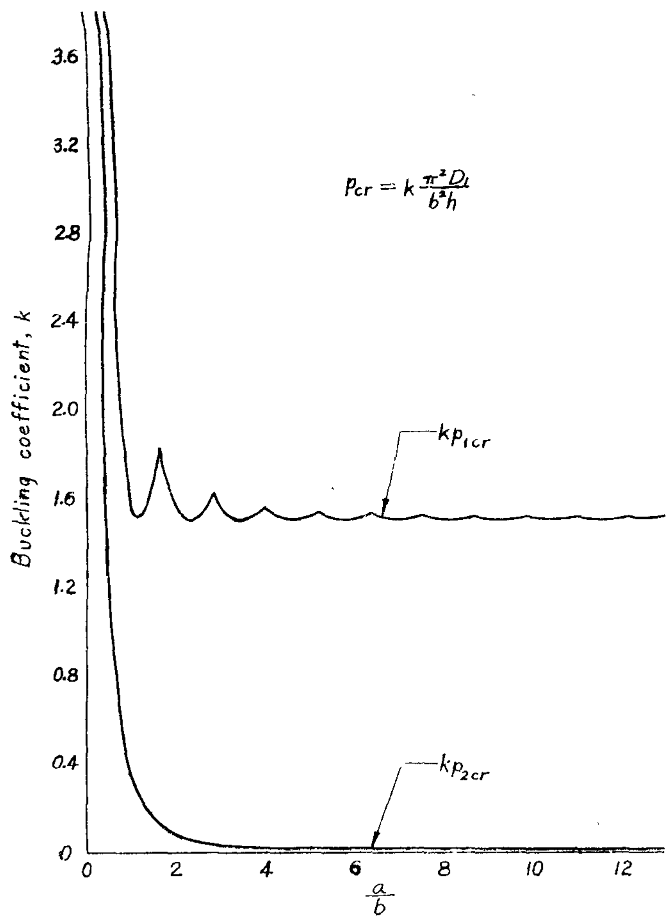


Fig. 5. Buckling coefficient for one edge clamped three edges simply supported plate.

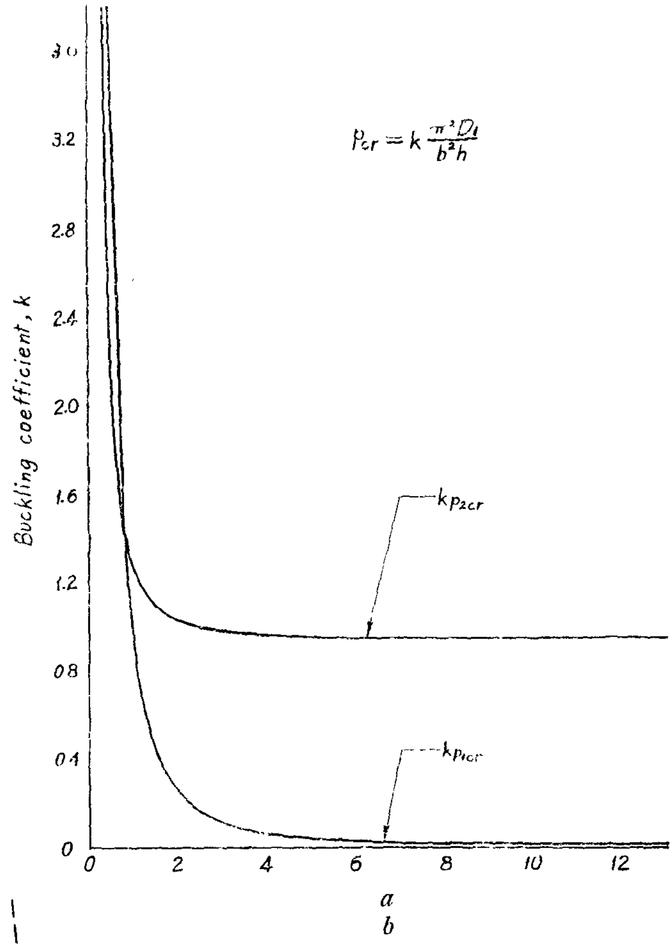


Fig. 6. Buckling coefficient for two oppesite edges clamped two opposite edges simply supported plate.

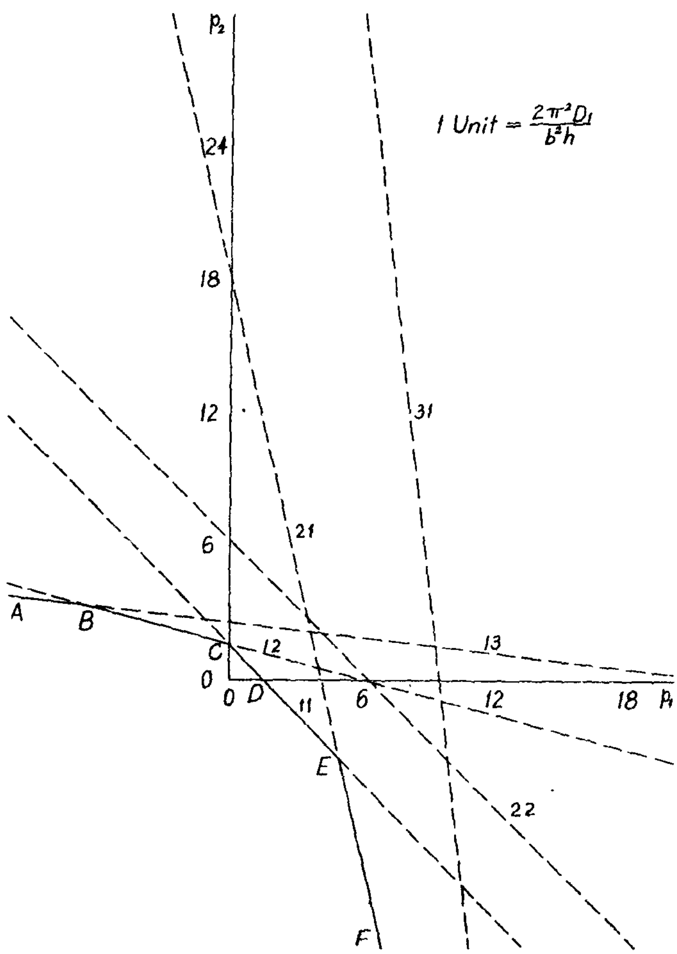


Fig. 7. Characteristic curves of p_1 and p_2 for various values of m and n.