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Excitation of Atmospheric Gravity Wave Modes on a Spherical Globe

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ABSTRACT

In this paper we investigate the excitation of gravity wave modes in a radially stratified non-isothermal spherical atmosphere. The exciting source can be time-space heating, mass injection or an external force. The problem is formulated in terms of a Green's function using eigen function expansion as done previously by Weston (1961). For a localized source, we find it convenient to express the solution as a sum of various spatial moments which are, in ascending order, total strength, bipolar moment, quadrupolar moment, etc. Making use of the addition theorem for noninteger degree Legendre functions, the solutions corresponding to the moments up to the fourth order have been obtained. It is found that the higher order moments of the source generally influence not only the wave amplitude but also its phase. This implies that, in general, the horizontal phase front is not necessarily orthogonal to the great circle linking the ground projections of the source and the observer. This certainly complicates the interpretation of the experimental data. By using a superposition integral, the monochromatic solution is further extended to the polychromatic case. It is found that a time dependent source can excite waves that are the sum of a forced oscillation and a free oscillation. If the source is localized, the forced oscillation will exist appreciably only near the source region. On the other hand, the free oscillations given by a discrete spectrum that satisfies a source-free dispersion relation can propagate a long distance away in response to the source.

Key Words: excitation, atmospheric gravity wave modes, Green's function, spatial moments, spherical globe

I. Introduction

The propagation of atmospheric waves around a spherical earth has been investigated since the early 1960s (Weston, 1961; Dikii, 1965; van Hulsteyn, 1965; Wait, 1969). In recent years the observational data of atmospheric waves in the aftermath of several severe geophysical events have been widely collected on a global scale and thoroughly analyzed (e.g., Liu *et al.*, 1982; Roberts *et al.*, 1982; Walker *et al.*, 1988; Williams *et al.*, 1988). Many of these observations have shown anisotropic excitation on a worldwide scale. For example, the gravity waves reported by Walker *et al.* (1988) for the Asia Pacific sector do not show equal strength for the American sector. In a theoretical study, Weston, among other authors, took into account both the sphericity of the earth and the atmospheric temperature profile. He employed an eigen function expansion approach to calculate the Green's function in a radially stratified atmosphere. The eigen functions are the solutions of the governing equation satisfying the boundary conditions at the earth surface (where the radial velocity of the air parcel must vanish) and infinity (toward which the wave amplitude must decrease exponentially). Each eigen function corresponds to a fully ducted propagating mode. The Green's function, expressed as a series of such eigen modes, satisfies the boundary conditions automatically. In the meantime, the ducted modes in a flat earth geometry have also been studied in great detail by many authors (e.g., Press and Harkrider, 1962; Harkrider, 1964; Friedman, 1966; Donn and Shaw, 1967; Francis, 1973). In this work we will use Weston's approach in our calculations and extend his results to excitation by anisotropic sources.

In most of the previous investigations of acoustic-gravity wave excitation problems using a flat earth geometry (Row, 1967; Liu and Yeh, 1971; Yeh and Liu, 1974) or a spherical earth geometry (Weston, 1961; Dikii, 1965; van Hulsteyn, 1965; Wait, 1969), the configuration of the source was assumed to be an isotropic singular point. This can be viewed as the first step toward investigating excitations by using a more general and more realistic case since an arbitrary source can be decomposed into an integral of point sources. If the source occupies a volume that is small in comparison with the region in which the excited wave propagates, a Taylor expansion of Green's function with respect to the source point can be used as is done for multipolar radiation of a charged body in electrodynamics. The moments of the source are found to carry important information about the strength, orientation and configuration of the source. Green's function itself can be viewed as the response to a zero order moment of the source. The higher order moments of the source give rise to a response with latitudinal and longitudinal dependences. The coexistence of higher moments in addition to the zero order moment would imply that the normal direction of the wave front is no longer aligned with the great circle path linking the observation point to the source point. This nonalignment greatly complicates the experimental data interpretation.

In handling the transient process of an excitation problem, Weston uses a time delay theorem in Fourier transform theory. This recipe, however, can only be effective in the non-dispersive case (Lamb wave mode). We address this problem with the insight of a linear system theory. In fact, as in the linear system theory, there are two kinds of singularities on the complex ω-plane: one originates from the frequency spectrum of the source, and the other originates from the "system," i.e., the atmosphere confined by the gravity force in a spherical globe. The former can be viewed as the forced oscillation directly driven by the source while the latter can be viewed as the free (source-less) oscillation. The forced oscillation can only last as long as the source is active. It exists only in an area that a sound wave starting from the source can cover during the source action, which usually takes a short period of time. The free oscillations, however, can last a much longer time and spread over a much wider area than can the forced oscillation. Since the atmosphere is assumed to be lossless, the singularities contributed by the "system" lie on the real ω-axis as discrete singularities. Thus, the free oscillation consists of a series of wave packets of various gravity modes with discrete horizontal-wavenumber-frequency spectra which correspond to a variety of Legendre functions of integral degrees. The discreteness of the horizontal-wavenumber-frequency spectra for free oscillations is intrinsically connected to the phenomenon that,after excitation, the wave energy propagates between the source region and its antipode.

II. Green's Function of a Spherical and Radially Stratified Atmosphere

We start with a set of basic linearized dynamic and thermodynamic equations for a radially stratified non-isothermal atmosphere with source terms included. Following the procedure of eigen function expansion employed by Weston (1961), we can write the Green's function for such a circumstance.

1. Basic Equations

The basic equations for linear perturbations in the atmosphere are as follows:

$$\frac{\partial s}{\partial t} + \mathbf{v} \cdot \nabla s_0 = \frac{Q}{T_0}, \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho_0 \mathbf{v}) = M, \tag{2}$$

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} + \nabla p + g\rho\hat{\mathbf{r}} = F. \tag{3}$$

As is clear in Eq. (2), the background atmosphere is assumed to be stationary. The effect of winds is known to give rise to the critical layer phenomenon (Booker and Bretherton, 1967; Brown and Stewartson, 1980). Therefore, the physics of the problem is already known even though the analysis has not been extended from Cartesian coordinates to spherical coordinates. In this paper, for simplicity, we choose to ignore the background wind effect, leaving it to later investigation. The variables in Eq. (1) through (3) are connected by two more equations: one comes from thermodynamic considerations, and the other expresses a hydrostatic equilibrium. They are

$$s = \frac{c_v}{p_0} p - \frac{c_p}{\rho_0} \rho \tag{4}$$

and

$$\frac{\partial p_0}{\partial r} = - g\rho_0, \tag{5}$$

where

- s, ρ, p are perturbations of entropy per unit mass, density and pressure, respectively;
- s_0, ρ_0, p_0 are the corresponding quantities for the background atmosphere;
- c_p and c_v are the specific heat at constant pressure and volume, respectively;

Q is the heat input per unit mass per unit time;
M is the mass injection rate;
$\boldsymbol{F}$ is the external force density;
$\hat{\mathbf{r}}$ is a unit radial vector against which a constant gravity g is applied.

The source terms can include various physical phenomena that may excite atmospheric acoustic-gravity waves such as Jole heating, nuclear explosion and the Lorentz force, etc.

Consider a temporal harmonic case in which the operator $\partial/\partial t$ becomes a scalar $j\omega$. After tedious calculation using Eq. (1) through (5), a scalar equation can be obtained:

$$L\psi=\xi. \tag{6}$$

In Equation (6), the wave function is $\psi=p/\sqrt{\rho_0}$. The acoustic-gravity wave operator in a spherical coordinate system has been previously obtained by Weston (1961) and is given by

$$L=\nabla^2-\frac{\partial^2}{\partial r^2}+\frac{\partial}{\partial r}(\zeta\frac{\partial}{\partial r})+q(r,\omega), \tag{7}$$

where the several symbols are defined by

$$q(r,\omega)=\frac{\partial(A\zeta)}{\partial r}+\frac{\omega^2}{r^2}-\zeta A^2;$$

$$A=\frac{g}{c^2}+\frac{1}{2\rho_0}\frac{\partial\rho_0}{\partial r};$$

$$\zeta=\frac{\omega^2}{\omega^2-\omega_b^2};$$

$\omega_b^2=(g/c_p)\partial s_0/\partial r$, square of local Brunt-Väisälä frequency.

The source term ξ on the right hand side of Eq. (6) is

$$\xi=\frac{1}{\sqrt{\rho_0}}\nabla\cdot F+\frac{1}{\sqrt{\rho_0}}\frac{\partial}{\partial r}[(\zeta-1)F_r]+\frac{\omega^2(\zeta-1)}{\sqrt{\rho_0}g}F_r$$

$$-\frac{j\omega M}{\sqrt{\rho_0}}-\frac{j\omega\zeta\sqrt{\rho_0}\,Q}{c_pT_0}-\frac{jg}{\sqrt{\rho_0}\omega c_p}\frac{\partial}{\partial r}(\frac{\zeta\rho_0 Q}{T_0}). \tag{8}$$

This source term was not given bu Weston. Once the wave function ψ is found, the velocity $\mathbf{v}$ of the air parcel can be expressed by

$$\mathbf{v}=\frac{j}{\sqrt{\rho_0}\omega}\left[\nabla\psi+(\zeta-1)\frac{\partial\psi}{\partial r}\hat{\mathbf{r}}+A\zeta\psi\hat{\mathbf{r}}\right]$$

$$-\frac{g\zeta Q}{\sqrt{\rho_0}\omega^2c_pT_0}\hat{\mathbf{r}}-\frac{j}{\rho_0\omega}[\mathbf{F}+(\zeta-1)F_r\hat{\mathbf{r}}], \tag{9}$$

where $F_r=\mathbf{F}\bullet\hat{\mathbf{r}}$. In particular, the radial velocity can be obtained by doting Eq. (9) by $\hat{\mathbf{r}}$ to yield

$$v_r=\mathbf{v}\cdot\hat{\mathbf{r}}=\frac{j\zeta}{\rho_0\omega}(\frac{\partial\psi}{\partial r}+A\psi)-\frac{g\zeta Q}{\omega^2c_pT_0}-\frac{j\zeta}{\rho_0\omega}F_r. \tag{10}$$

To solve the excitation problem of Eq. (6), we take the Green's function approach. Using this approach, the solution to Eq. (6) is expressed as

$$\psi(\mathbf{r})=\int_\Omega\xi(\mathbf{r}_s)G(\mathbf{r},\mathbf{r}_s)d^3\mathbf{r}s. \tag{11}$$

Here, the Green's function G is the solution to the following equation

$$LG=\delta(\mathbf{r}-\mathbf{r}_s), \tag{12}$$

where the wave operator L is given by Eq. (7).

In Eq. (11), the integration over the source coordinates $\mathbf{r}_s$ covers the volume Ω. We consider the case in which the source volume Ω is small in comparison with the wave propagation region. In this case we can expand the Green function $G(\mathbf{r},\mathbf{r}_s)$ with respect to $\mathbf{r}_s$ in the form of a Taylor series around a certain point $\mathbf{r}_0$ inside Ω. As an approximation, we truncate the series after the second order. The wave function then becomes, approximately,

$$\psi(\mathbf{r})=B_0G(\mathbf{r},\mathbf{r}_0)+\mathbf{B}_b$$

$$\cdot\nabla_sG(\mathbf{r},\mathbf{r}_s)\big|_{r_s=r_0}+\frac{1}{2}\overleftrightarrow{\mathbf{B}}_q:\nabla_s\nabla_sG(\mathbf{r},\mathbf{r}_s)\big|_{r_s=r_0} \tag{13}$$

The coefficients B_0, $\mathbf{B}_b$ and $\overleftrightarrow{\mathbf{B}}_q$ are the total strength, bipolar and quadrupolar moments of the source. They all reflect the structure and orientation of the source. They are defined by

$$B_0=\int_\Omega\xi d^3\eta,\quad \mathbf{B}_b=\int_\Omega\xi\boldsymbol{\eta}d^3\eta,$$

$$\overleftrightarrow{\mathbf{B}}_q=\int_\Omega\xi\boldsymbol{\eta}\boldsymbol{\eta}d^3\eta, \tag{14}$$

where $\eta=\mathbf{r}_s-\mathbf{r}_0$. It is easy to show that B_0, B_b and $\overleftrightarrow{\mathbf{B}}_q$ are the 0-th, 1-st and 2-nd coefficients of a Taylor expansion of the spectrum of the source density ξ.

2. Green's Function

To simplify the expressions, we assume $\mathbf{r}_s = r_0\hat{\mathbf{z}}$; i.e., the source point is on the polar axis for which the Green's function has an azimuthal symmetry. In this case, Eq. (12) can be written as

$$LG = \delta(\mathbf{r} - r_0\hat{\mathbf{z}}). \tag{15}$$

Under the approximation that the atmosphere is thin in comparison with the earth radius, the operator L becomes separable. In this case, the solution to Eq. (15) can be approximately written as

$$G(\mathbf{r}, \mathbf{r}_0) = \sum_i \frac{R_i(r_0)R_i(r)P\mu_i(-\cos\theta)}{4W_i \sin\mu_i\pi}, \tag{16}$$

where θ is the angle between the two position vectors $\mathbf{r}$ and $\mathbf{r}_0$. The function P_μ is a Legendre function of degree μ, and the function R_i is a solution to the following eigen equation

$$[\frac{d}{dr}(\zeta\frac{d}{dr}) + q(r, \omega)]R_i - \lambda_i R_i = 0 \tag{17}$$

with a corresponding eigen value λ_i. The degree and the eigen value are related by $\mu_i(\mu_i+1)=\lambda_i r_0^2$. The eigen function R_i represents the ith vertical distribution of a perfectly ducted mode caused by the temperature profile of the lower atmosphere. There is a constraint between λ_i and ω for every mode. This constraint stems from the boundary conditions ar $r=a$ and $r\to\infty$. The relation between λ_i and ω may be called a dispersion relation and is expressed as the solution to the following implicit relation:

$$\Lambda(\omega, \lambda)=R'(a+h_0;\ 1, -A_0;\ \omega, \lambda)$$

$$\sqrt{Q(a+h_o;\omega,\lambda)} \bullet R(a+h_0;\ 1, -A_0;\ \omega, \lambda)=0. \tag{18}$$

This implicit relation is obtained from the upper boundary as $r\to\infty$ by requiring that wave function decay exponentially in the thermosphere. The altitude h_0 is somewhere in the thermosphere where the atmosphere becomes isothermal. The eigen function $R(r;\ R_0, R_0';\ \omega, \lambda)$ is a solution to Eq. (17) satisfying the boundary conditions

$$R(a;\ R_0, R_0';\ \omega, \lambda)=R_0 \tag{19}$$

and

$$R'(a;\ R_0, R_0';\ \omega, \lambda)=R_0'. \tag{20}$$

The vanishing of v_r at $r=a$ links R_0 and R_0' by the relation $R_0' + A_0R_0=0$ where $A_0=A$ at $r=a$. For a given ω, there are many values of λ_i satisfying Eq. (18), each of which belongs to a specific fully ducted mode.

The Green's function expression Eq. (16) shows that the relative strength of a mode depends on the height at which the source is placed. In fact, every eigen mode is represented by a standing wave distribution along the radial direction. If the source is at a height where the eigen mode assumes a crest value, this mode will be highly excited since the coefficient of this mode in Eq. (16), $R_i(r_0)/W_i$ is large whereas if the source is at a height at which another mode has a node, this other mode is not likely to be excited at all since the coefficient will be zero.

III. Anisotropic Excitation of Gravity Wave Modes

We have seen from Eq. (13) that the anisotropic excitation problem can be solved by calculating various orders of the gradient of the Green's function with respect to the source position vector. To do so, we need to shift the source point from the polar axis of the spherical coordinate system $r_0\hat{\mathbf{z}}$ to a new position, $\mathbf{r}_s=r_s\hat{r}_s$. The non-integer-degree Legendre function has the following addition formula that will be useful for this purpose:

$$\begin{aligned} &P_\mu(\cos\theta_{12}) \\ &= P_\mu(\cos\theta_1)P_\mu(\cos\theta_2) \\ &\quad + 2\sum_{m=1}^{\infty}\frac{\Gamma(\mu-m+1)}{\Gamma(\mu+m+1)}P_\mu^m(\cos\theta_1) \\ &\quad \bullet P_\mu^m(\cos\theta_2)\cos m(\phi_1-\phi_2), \end{aligned} \tag{21}$$

where θ_{12} is the polar angle of point 2 relative to point 1, i.e.,

$$\cos\theta_{12}=\cos\theta_1\ \cos\theta_2+\sin\theta_1\ \sin\theta_2\ \cos(\phi_1-\phi_2).$$

The function P_μ^m is the associated Legendre function of degree μ of order m. It can be proven from Eq. (21) that

$$\begin{aligned} &P_\mu(-\cos\theta_{12}) \\ &= P_\mu(-\cos\theta_1)P_\mu(\cos\theta_2) \\ &\quad + 2\sum_{m=1}^{\infty}(-1)^m\frac{\Gamma(\mu-m+1)}{\Gamma(\mu+m+1)}P_\mu^m(-\cos\theta_1) \end{aligned}$$

$$\bullet P_\mu^m(\cos\theta_2)\cos m(\phi_1-\phi_2). \quad \textbf{(22)}$$

The stage is now set for calculating the gradients needed in Eq. (13). Substituting the addition formula Eq. (22) into the Green's function Eq. (16), and making use of many formulas applicable to Legendre functions (Abramowitz and Stegun, 1964), the following results are obtained:

$$\nabla_s G|_{r_s=r_0}$$

$$=\sum_\iota \frac{R_\iota(r)}{4W_\iota \sin\mu_\iota\pi}\left[\frac{\mu_\iota(\mu_\iota+1)R_\iota(r_0)}{r_0}P^1_{\mu_\iota}(-\cos\theta)\,\hat{\tau}\right.$$

$$\left.+R_\iota'(r_0)P_{\mu_\iota}(-\cos\theta)\,\hat{\mathbf{z}}\right] \quad \textbf{(23)}$$

$$\nabla_s\nabla_s G|_{r_s=r_0}$$

$$=\sum_\iota \frac{R_\iota(r)}{4W_\iota\sin\mu_\iota\pi}\left\{\left[\frac{R_i'(r_0)r_s-\mu_\iota(\mu_\iota+1)R_\iota(r_0)}{r_0^2}\right.\right.$$

$$\left.\bullet P_{\mu_\iota}(-\cos\theta)-\frac{R_i(r_0)}{4r_0^2}P^2_{\mu_i}(-\cos\theta)\right]\overline{\mathbf{I}}_2$$

$$+\frac{R_\iota(r_0)}{2r_0^2}P_0^2(-\cos\theta)\,\hat{\tau}\hat{\tau}+R_i''(r_0)P_{\mu_\iota}(-\cos\theta)\,\hat{\mathbf{z}}\hat{\mathbf{z}}$$

$$\left.+\frac{R_\iota'(r_0)r_0-R_s(r_0)}{2r_0^2}P^1_{\mu_\iota}(-\cos\theta)(\hat{\tau}\hat{\mathbf{z}}+\hat{\mathbf{z}}\hat{\tau})\right\}, \quad \textbf{(24)}$$

where the unit vector $\hat{\tau}$ is in a direction tangent to a great cycle path on the earth surface formed by projections of the source point to the observation point, and $\overline{\mathbf{I}}_2$ is a 2-D unit tensor in the horizontal plane.

We denote the bipolar and quadrupolar moments of the source as follows:

$$\mathbf{B}_b=B_{bx}\hat{\mathbf{x}}+B_{by}\hat{\mathbf{y}}+B_{bz}\hat{\mathbf{z}} \quad \textbf{(25)}$$

$$\overline{\mathbf{B}}_q=B_{qx}\hat{\mathbf{x}}\hat{\mathbf{x}}+B_{qy}\hat{\mathbf{y}}\hat{\mathbf{y}}+B_{qxy}(\hat{\mathbf{x}}\hat{\mathbf{y}}+\hat{\mathbf{y}}\hat{\mathbf{x}})+B_{qxz}(\hat{\mathbf{x}}\hat{\mathbf{z}}+\hat{\mathbf{z}}\hat{\mathbf{x}})$$

$$+B_{qyz}(\hat{\mathbf{y}}\hat{\mathbf{z}}+\hat{\mathbf{z}}\hat{\mathbf{y}})+B_{qz}\hat{\mathbf{z}}\hat{\mathbf{z}}, \quad \textbf{(26)}$$

where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ are unit vectors of x and y axes which are in the horizontal plane with the source point at the origin. Expressed in terms of the components of these moments, the approximate Green's function Eq. (13) is found to be

$$\psi(\mathbf{r})$$

$$=\sum_\iota\frac{R_\iota(r)}{4W_\iota\sin\mu_\iota\pi}\Big(\Big\{B_0R_\iota(r_0)+B_{bz}R_\iota'(r_0)+\tfrac{1}{2}B_{qz}R_\iota''(r_o)$$

$$+\frac{(B_{qx}+B_{qy})[R_\iota'(r_0)r_0-\mu_\iota(\mu_\iota+1)R_\iota(r_0)]}{2r_0^2}\Big\}$$

$$\bullet P_{\mu_\iota}(-\cos\theta)+\Big\{\Big[\frac{\mu_\iota(\mu_\iota+1)R_\iota(r_0)}{r_0}B_{bx}$$

$$+\frac{R_\iota'(r_0)r_0-R_\iota(r_0)}{2r_0^2}B_{qxz}\Big]\cos\phi$$

$$+\Big[\frac{\mu_\iota(\mu_\iota+1)R_\iota(r_0)}{r_0}B_{by}+\frac{R_\iota'(r_0)r_0-R_\iota(r_0)}{2r_0^2}B_{qyz}\Big]$$

$$\bullet\cos\phi\Big\}P^1_{\mu_\iota}(-\cos\theta)+\frac{R_\iota(r_0)}{4r_0^2}\Big[\frac{(B_{qx}-B_{qy})}{2}\cos 2\phi$$

$$+B_{qxy}\sin 2\phi\Big]P^2_{\mu_\iota}(-\cos\theta)\Big), \quad \textbf{(27)}$$

where $\phi=\angle(\hat{\tau},\hat{x})$.

From Eq. (27), we see that the horizontal components of a bipolar moment contribute to the cosϕ and sinϕ azimuthal dependence while the horizontal components of a quadrupolar moment contribute to the cos2ϕ and sin2ϕ azimuthal dependence of the excited modes. Furthermore, the coexistence of azimuthally symmetric terms and non-symmetric terms, especially the zero and first order terms, produces both amplitude and phase gradients in the ϕ direction. This azimuthal phase gradient makes horizontal wave propagation direction non-orthogonal to a great circle path linking ground projections of the source and observation point. The asymptotic approximation of an associated Legendre function (Abrahamowitz and Stegun, 1964) gives

$$P_\mu^m(\cos\theta)\approx\frac{\Gamma(\mu+m+1)}{\Gamma(\mu+3/2)}\sqrt{\frac{2}{\pi\sin\theta}}$$

$$\bullet\cos\left[(\mu+\tfrac{1}{2})\theta-\tfrac{\pi}{4}+\tfrac{m\pi}{2}\right]. \quad \textbf{(28)}$$

Thus,

$$aP_\mu(-\cos\theta)+b\cos\phi\bullet P^1_\mu(-\cos\theta)$$

$$\approx A\cos[(\mu+1/2)\theta-\delta]+B\cos\phi\bullet\cos[(\mu+1/2)\theta-\pi/2-\delta]$$

$$\to[A-jB\cos\phi]\,e^{-j(\mu+1/2)\theta-\delta}, \quad \textbf{(29)}$$

where

$$A=\frac{\Gamma(\mu+1)}{\Gamma(\mu+3/2)}\sqrt{\frac{2}{\pi\sin\theta}}\bullet a,$$

$$B=\frac{\Gamma(\mu+2)}{\Gamma(\mu+3/2)}\sqrt{\frac{2}{\pi\sin\theta}}\bullet b, \qquad \delta=\mu\pi.$$

The factor $A-jB\cos\phi$ produces not only an amplitude but also a phase modulation along the ϕ direction. This is depicted in Fig. 1 when the wave front given by Eq. (29) with B/A=0.5 is shown. The possible existence of ϕ variations greatly complicates the data interpretation in general.

This phenomenon may help diagnose the nature of the source by using the data acquired in remote areas situated in all directions from the source. For example, the micro barograph data recorded in Japan in the aftermath of the eruption of Mount St. Helens on May 18, 1980 shows that the horizontal propagation direction is roughly along the great circle path (Liu *et al.*, 1981). That means the bipolar moment of the source is probably nearly vertical to the earth surface.

Amplitude modulation along the azimuthal direction caused by higher order moments of the source makes the wave strength vary along the ϕ direction. This is similar to antenna patterns in antenna theory. Figure 2 gives an example of the amplitude modulation patterns.

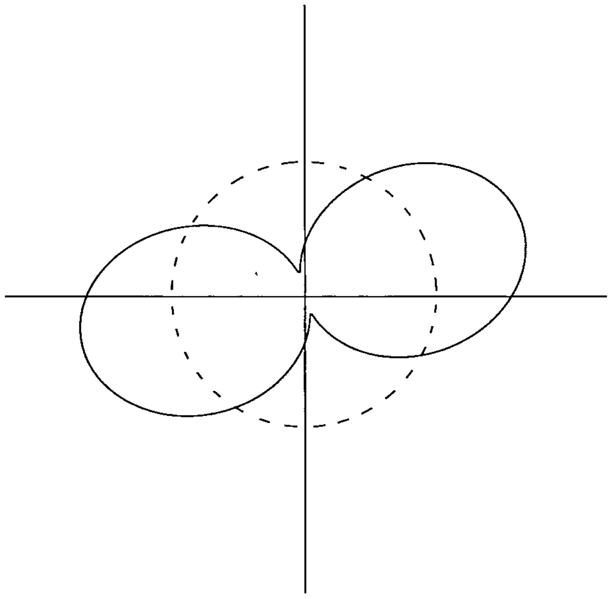

Fig. 2. The azimuthal wave strength pattern of a wave mode $P_n(\cos\theta)+0.7\bullet P_n^2(\cos\theta)\bullet\cos(2\phi-30°)$.

IV. Transient Propagation Process of Anisotropic Excitation of Gravity Wave Modes

The results obtained in previous sections apply to monochromatic waves excited by a monochromatic source, By using a superposition integral, these results can be extended to problems involving polychromatic excitations. In fact, the source moments used in the last section can be viewed as Fourier components of a polychromatic source. According to the superposition principle, the response of the perturbed atmosphere to a polychromatic source can be obtained by integrating the monochromatic components of the response function as

$$\psi(\mathbf{r},t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\psi(\mathbf{r},\omega)\,e^{j\omega t}d\omega$$

$$=\sum_i\int_{-\infty}^{\infty}\frac{F_i(r,\theta,\phi;r_0,\omega)}{\sin\mu(\omega)\pi}\,e^{j\omega t}d\omega, \tag{30}$$

where from Eq. (24) F_i is given by

$$F_i=(r,\theta,\phi;r_0,\omega)$$

$$=\frac{R_i(r)}{4W_i}\{[B_0R_i(r_0)+B_{bz}R_i'(r_0)+\frac{1}{2}B_{qz}R''(r_0)$$

$$+\frac{(B_{qx}+B_{qy})[R_i'(r_0)r_0-\mu_i(\mu_1+1)R_i(r_0)]}{2r_0^2}]$$

$$\bullet P_{\mu_i}(-\cos\theta)+[[\frac{\mu_i(\mu_i+1)R_i(r_0)}{r_0}B_{bx}$$

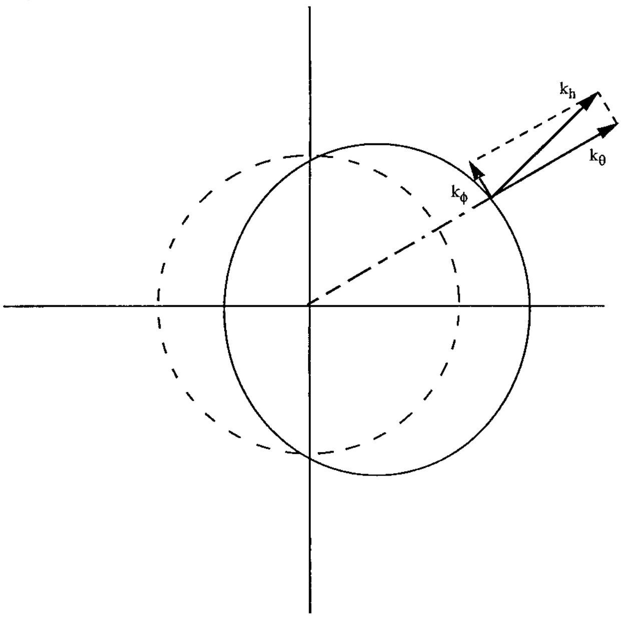

Fig. 1. The wave front of the wave mode $P_n(\cos\theta)+\beta\bullet P_n^1(\cos\theta)>\cos\phi$, where n=40, β=–0.5 There is an azimuthal phase gradient k_ϕ to make the horizontal wave propagation direction k_h off the great circle path direction k_θ.

$$+\frac{R_i'(r_0)\,r_0-R_i(r_0)}{2r_0^2}B_{qxz}]\cos\phi$$

$$+[\frac{\mu_i(\mu_i+1)\dot{R}_i(r_0)}{r_0}B_{by}+\frac{R_i'(r_0)\,r_0-R_i(r_0)}{2r_0^2}B_{qyz}]$$

$$\bullet\sin\phi]P^1_{\mu_i}(-\cos\theta)+\frac{R_i(r_0)}{4r_0^2}[\frac{B_{qx}-B_{qy}}{2}\cos 2\phi$$

$$+B_{qxy}\sin 2\phi]P^2_{\mu_i}(-\cos\theta)\}.$$

Since in general B_0, $\mathbf{B}_b$, $\overleftrightarrow{\mathbf{B}}_q$, $R_i(r)$, $R_i(r_0)$ and μ are all the functions of ω, it is very difficult to obtain an analytic result for Eq. (30) except for some special cases and under certain assumptions or approximations. One such case is the Lamb wave mode, which will be considered in the following. For this mode and with the mode number i dropped hereafter, the radial function $R(r)$ is independent of ω and $\mu = r_0\omega/c - 1/2$ for large $|\mu|$.

In pursuing solutions, notice that the factor sin $\mu(\omega)\pi$ in the denominator of the last expression of Eq. (30) has a series of zeroes when $\mu(\omega)$ is equal to an integer n. Since the atmosphere is assumed to be lossless and the modes considered are perfectly ducted, the equation $\mu(\omega)=n$ has a real root ω_n. Thus, there exists a series of poles on the real ω axis. To satisfy the principle of causality, the poles of $F(r, \theta, \phi; r_0, \omega)$ must be in the upper half ω-plane. This can be satisfied by an integration path along a straight-line parallel to the real ω-axis in the lower ω-plane (Fig. 3). Thus, the integration of Eq. (30) consists of contributions from residues of two groups of poles in the ω-plane; one group comes from the poles of $F(r, \theta, \phi; r_0, \omega)$, and the other comes from zeroes of sin $\mu(\omega)\pi$. The residues from the poles of F can be identified as the forced oscillations while those from zeroes of sin$\mu\pi$

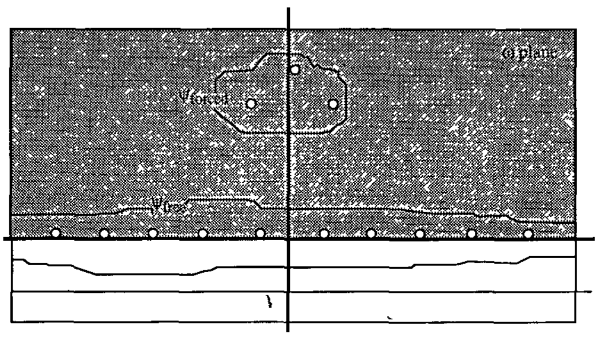

Fig. 3. A graph depicting the singularities contributed by both sin$[\mu(\omega)\pi]$ and F in the complex ω-plane.

can be identified as the free oscillations. Writing them out explicitly, we have

$$\psi = \psi_{\text{forced}} + \psi_{\text{free}}, \tag{31}$$

where

$$\psi_{\text{free}} = \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\partial\omega}{\partial\mu}\bigg|_{\omega=\omega_n}(-1)^n F(r,\theta,\phi;r_0,\omega_n). \tag{32}$$

In $F(r, \theta, \phi; r_0, \omega_n)$, the degree μ of the Legendre functions must necessarily become an integer n. Notice that $P_n^m(-x) = (-1)^n P_n^m(x)$; therefore,

$$(-1)^n F(r,\theta,\phi;r_0,\omega_n)$$

$$=\frac{R(r)}{4W}\{(B_0R(r_0)+B_{bz}R'(r_0)+\tfrac{1}{2}B_{qz}R''(r_0)$$

$$+\frac{(B_{qx}+B_{qy})[R'\,r_0)\,r_0-\mu(\mu+1)R(r_0)]}{2r_0^2})P_n(\cos\theta)$$

$$+[\frac{\mu(\mu+1)R(r_0)}{r_0}B_{bx}+\frac{R'(r_0)\,r_0-R(r_0)}{2r_0^2}B_{qxz}]\cos\phi$$

$$+[\frac{\mu(\mu+1)R(r_0)}{r_0}B_{by}+\frac{R'(r_0)\,r_0-R(r_0)}{2r_0^2}B_{qyz}]$$

$$\bullet\sin\phi)P_n^1(\cos\theta)+\frac{R(r_0)}{4r_0^2}[\frac{(B_{qx}-B_{qy})}{2}\cos 2\phi$$

$$+B_{qxy}\sin 2\phi]P_n^2(\cos\theta)\}. \tag{33}$$

This solution has no singularities for any values of r, θ and ϕ and, therefore, is a solution of Eq. (6) when $\xi=0$. For this reason, it represents the free oscillations excited by the source.

The contribution from the poles of $F(r, \theta, \phi; r_0, \omega)$ represents the forced oscillations directly driven by a source:

$$\psi_{\text{forced}} = \sum_j res\{\frac{F(r,\theta,\phi;r_0,\omega)}{\sin\mu(\omega)\pi}e^{j\omega t}\}\bigg|_{\omega=\omega_j} \tag{34}$$

To simplify the calculations, we assume the frequency spectrum of all source moments to have the following form:

$$A(\omega) = \frac{\tau}{1+j\omega\tau}. \tag{35}$$

That is, $F(r, \theta, \phi; r_0, \omega)$ has a single first order pole at $\omega=j/\tau$. For Lamb waves, $\mu+1/2 \approx r_0\omega/c$ for $|\mu| >> 1$.

Consequently, there is no dispersion, and R and W are independent of ω.

Using the asymptotic expression Eq. (28), the forced Lamb wave solution becomes

$$\psi_{\text{forced}} = e^{-\sigma\theta} \cdot \frac{R(r)}{2W_t\sqrt{\sin\theta}} \{ [B_0 R(r_0)$$

$$+ B_{bz} R'(r_0) + \frac{1}{2} B_{qz} R''(r_0)$$

$$+ \frac{(B_{qz}+B_{qy})[R'(r_0)r_0 - \mu(\mu+1)R(r_0)]}{2r_0^2}]$$

$$\cdot \frac{\Gamma(1/2+j\sigma)}{\Gamma(1+j\sigma)} e^{j\pi/4} + [(\frac{\mu(\mu+1)R(r_0)}{r_0} B_{bx}$$

$$+ \frac{R'(r_0)r_0 - R(r_0)}{2r_0^2} B_{qxz})\cos\phi$$

$$+ (\frac{\mu(\mu+1)R(r_0)}{r_0} B_{by} + \frac{R'(r_0)r_0 - R(r_0)}{2r_0^2}$$

$$\cdot B_{qyz})\sin\phi] \frac{\Gamma(3/2+j\sigma)}{\Gamma(1+j\sigma)} e^{-j\pi/4}$$

$$+ \frac{R(r_0)}{4r_0^2} [\frac{(B_{qz}-B_{qy})}{2} \cos 2\phi + B_{qxy}] \sin 2\phi)$$

$$\cdot \frac{\Gamma(5/2+j\sigma)}{\Gamma(1+j\sigma)} e^{-j3\pi/4} \}, \tag{36}$$

where $\sigma = r_0/c\tau$. We notice that this term decreases exponentially at a rate of σ as θ increases. The quantity σ is a large number since τ can be viewed as a measure of the source action time which is usually short, making $c\tau << a < r_0$. As a consequence the forced oscillation contributes significantly only in the vicinity of the source during source action. In remote areas in the aftermath of the source action, the free oscillation terms will, therefore, dominate in response to the source action.

The free oscillation term ψ_{free} constitutes a discrete spectrum of monochromatic waves. Notice that every $P_n^m(\cos\theta)$ represents a standing oscillation along the θ-direction. This standing oscillation can be further decomposed into two opposite traveling waves between the source and its antipode. By using the Poisson formula (Papoulis, 1962), the global atmospheric response can be interpreted as a process whereby wave energy of discrete spectrum propagates back-and-forth between the source point and its antipode. This can greatly complicate the experimental data interpretation, especially in the backward ray tracing technique in locating the source as cautioned by the authors in an earlier paper (Dong and Yeh, 1993).

V. Conclusions

From the investigations conducted in previous sections, we can draw the following conclusions.

Atmospheric gravity wave modes can be excited by various physical causes ranging from localized sudden heating, mass injection, or external forces like the Lorentz force. The radial part of the Green's function as given by the eigen modes in Eq. (16) represents standing waves in the radial direction. For excitation of a particular mode, the source must possess substantial energy near the crest of that mode. Thus, a localized source will excite only those modes that have substantial amplitude at the source height. The vertical distribution of energy profiles of various modes has been computed (e.g. Francis, 1973; Maeda, 1982, 1985). The region where substantial energy exists is known as the duct. Thus, to excite a particular mode, the source of that mode must be placed inside the duct. The excited wave field is determined not only by the total strength of the source (the integral of source density over the volume in which the source resides, or the 0-th order moment of the source), but also by its higher order moments as well. The higher order moments determine the azimuthal variations of wave amplitude and phase. The coexistence of the zero order moment and horizontal components of bipolar moment of the source in general makes the horizontal wave propagation direction non-orthogonal to the great circle path linking the source and the observer. This can be used to good advantage as a means of diagnosing the nature of the source. This also means that the widely assumed plane wave fronts that have been used in experimental data analysis must be examined carefully for validity on a case by case basis.

The transient response of the atmosphere to a severe event like earthquake, volcanic eruption or nuclear explosion consists of two parts. The first part, called source forced oscillation, is directly linked to the source action. The source forced oscillation lasts for as long as the source is active. The range within which the forced oscillation is appreciable is about the distance the sound covers during the source action, which is much smaller than the earth's radius. The second part, called free oscillations, can last much longer than the source does and may propagate over long distances. The free oscillation is a solution of the source-less equation and has a discrete frequency-horizontal-wave-number spectrum. The discrete spectrum corresponds to the back-and-forth movement of wave energy between the source and its antipode on the earth's surface.

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聲重波在地球大氣中的激發

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摘　要

本文探討了在非同溫大氣中聲重波的激發。激發的波源可以是時空熱能、物質的注射，或外來動力。爲探討此問題，我們應用了格林函數，並以特征函數（eigen function）展開。若波源爲局部的，問題的解可用空間階矩（spatial moments）的和指示，並只取幾個低階矩爲近似值。在本文中我們求出最高四限階矩並發現高限階矩不但會影響波的振幅，且對波的相位亦有所改變。這個結果對今後實驗資料之分析提出謹慎的信號。我們更用重疊積分（superposition integral）將單色結果推廣至多色結果。因而發現所激發的波可分爲二類。一類爲被迫振盪（forced oscillation）。若波源爲局部的，被迫振盪僅存在於波源附近。第二類爲自由振盪（free oscillation）。自由振盪波滿足無源的色散關係（dispersion relation）。可長距離的傳播，甚至傳到源在球面上的對蹠點（antipode）。