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Questions during Problem Solving with Dynamic Geometric Software and Understanding Problem Situations

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ABSTRACT

Previous research has suggested that asking "why" questions is important for mathematical problem solving. The purpose of this paper was to analyze students' "why" questions which arose during their problem-solving processes and to provide insights into the nature of such questions. For this purpose, we analyzed the processes of solving two problems by a pair of ninth graders, for which they tackled geometry problems using dynamic geometry software. They asked or were prompted to ask "why" questions several times. The analysis showed that as the students explored the problem situation and deepened their understanding of the situation, they related their "why" questions to the mechanism of the situation and pursued the questions more seriously. This implies that asking genuine "why" questions is supported by the solvers' understanding of the problem situation. This also implies that it is important for students to explore problem situations to some extent before they are capable of asking genuine questions.

Key Words: mathematical problem solving, questions, thought process, understanding, problem situation

I. Introduction

Charles & Lester (1982: p. 5) defined a problem as follows: "A problem is a task for which: (1) The person confronting it wants or needs to find a solution; (2) The person has no readily available procedure for finding the solution; (3) The person must make an attempt to find a solution." The first component is concerned with emotional aspects or intrinsic motivation with respect to the problem solver. Sohma (1997) stated that he wanted his students to experience a feeling of "why?" so that they would be motivated to solve mathematical problems. In fact, when a person has a feeling of "why?" concerning a problem, he/she may want to know the reason for that "why?" and investigate the problem further. For example, when a person confronts a strange phenomenon in a problem situation and has a question, such as "why do such things happen?" he/she may have a desire to find some reasons to answer this question. Having such questions changes tasks given to students into problems they truly desire to resolve. Asking "why?" is important especially in school mathematics. Hanna (1995) distinguished two types of proofs, "proofs that prove" and "proofs that explain." While the former deals with "what is true?" the latter deals with "why is it true?" (see also Hersh (1997: pp. 59-61)). She emphasized the importance of "proofs that explain" in school mathematics. Since almost all statements concerning proof problems in school mathematics have already been proven by others, proof problems may become worthy of being explored when students desire to explain why such statements are true. This implies that students' feelings of "why?" are critical in turning those problems into genuine issues.

According to Stacey & Scott (2000), having a "why" question not only provides motivation for problem solving, but also serves as a key to successful mathematical problem solving. They examined ways in which students tried examples while solving a problem, where they were asked to find numbers to satisfy a certain condition. They found that orientation toward a deeper structure is a key component for the effective use of examples in developing solutions. Here, they used the term "surface structure" to indicate what is true about a mathematical situation and the term "deep structure" to indicate why it is true. Thus, their finding implies that asking "why" questions during problem solving and being oriented to the deep structure is key to successful problem solving.

The previous research mentioned above suggests that asking "why" questions is important for various aspects of mathematical problem solving. In spite of this importance, there has been little research on "why" questions themselves. Exploring how such questions are asked during problem solving processes would be helpful in understanding these processes of students.

The purpose of this paper is to analyze students' "why" questions during their problem-solving processes to provide insights into the nature of such questions. For this purpose, we analyze the processes of solving two problems by a pair of ninth graders who were tackling geometry problems during which they asked several "why" questions. Although the questions asked in one process were similar to each other, their tones and intensities (i.e., the extent a student stuck with a question) seemed to slightly differ. Such differences and changes in the questions are examined based on the protocols of the solution processes. This examination illustrates a factor that is important for asking "why" questions during mathematical problem solving.

II. Data Gathering

The data analyzed in the remainder of this paper were gathered during research mainly implemented by the second author. The aim of this research was to explore roles of dynamic geometry software (Goldenberg & Cuoco, 1998) in developing ideas to prove geometry problems, and to explore the effects of teacher interventions which were planned in advance (Fukuzawa, 2001). Cabri Geometry was used as the dynamic geometry software in this research [1]. Subjects consisted two pairs of Japanese ninth graders, the last grade of junior high school, and each pair separately participated in five sessions and a follow-up interview. Since the participants had never used this software, instructions in its basic commands and exercises for the construction of basic figures were given during the initial three sessions. After this introduction, two problem-solving sessions were implemented, and the students tackled one geometry problem using Cabri Geometry in each session. The problems used in these problem-solving sessions are described below.

Problem 1. Construct a quadrilateral ABCD. Let the midpoint of side AB be P, the midpoint of BC be Q, the midpoint of CD be R, and the midpoint of DA be S. When connecting these midpoints, what kind of quadrangle does PQRS become? (Kakihana & Shimizu (1997); modified by the authors; see the figures in Section III)

Problem 2. There is a triangle ABC. Construct an equilateral triangle BAD on the side opposite to ΔABC. Construct an equilateral triangle ACE on the side opposite to ΔABC. Construct an equilateral triangle BCF on the same side as ΔABC. What kind of quadrangle does ADFE become? (Nohda & Nakayama (1996); modified by the authors; see the figures in Section IV)

The entire problem-solving sessions were recorded with an audiotape recorder and two video cameras. One video camera recorded the computer screen and the other recorded the students. Protocols were made based on these audio and video records. Using video records of the screen, we could include the tracings of their dragging operations and the order in which they adopted commands during their problem-solving activities in our protocols.

In this paper, we examine only one pair of male students, Nogawa and Yamada (pseudonyms), and analyze each of their problem-solving processes to explore how their questions changed. A tendency similar to what is presented in this paper was observed in the problem-solving processes of the other pair. However, Nogawa and Yamada were selected for illustration because changes in their "why" questions were more clearly observable.

III. Questions Arising while Solving Problem 1

1. Outline of Their Solution for Problem 1

(i) The students constructed rectangle ABCD. Nogawa operated the software (as he did throughout the process of solving the problem). Although they used the Cabri Geometry software, they constructed the rectangle solely by resorting to visual judgment, and did not use commands like "perpendicular line" or "parallel line" which were available. They then connected the midpoints of the four sides to make PQRS. They drew PR and QS, made the intersection point O, and measured the lengths of PO, QO, RO, and SO.[2] They said that PQRS seemed to be a rhombus, and measured the lengths of PQ, QR, RS, and SP. The students also mentioned that PQRS might be a parallelogram. They measured ∠RSQ saying that the opposite angles were equal, and then they used the "parallel?" command, which checks whether or not two lines are parallel, to determine that PS was parallel to QR. They also tried to measure the area of PQRS but failed to do so.

(ii) When they dragged the point D to arrange the lengths of the sides, ABCD was out of the rectangle, and so they thought that it was better to call PQRS a parallelogram. They measured ∠PSQ, ∠PQS, and ∠RQS, and found that opposite angles were equal and remained equal even when dragging C

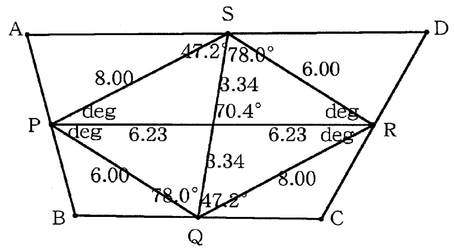

Fig. 1. Diagram used when the students concluded that PQRS was a parallelogram in (iii) ("deg" indicates that those angles were measured, but their exact values could not be read from the VTR data).

and D. The students also mentioned, according to the measurement values on the screen, that the opposite sides were equal and that they were parallel to each other since the "alternate interior angles" were equal.

(iii) The students wanted to know the area of PQRS which they thought "crucial evidence." They tried to change the lengths of the sides of PQRS so that they could easily calculate the area using a mathematical formula. After giving up that attempt, they measured ∠SPR, ∠QPR, ∠SRP, and ∠QRP. They confirmed that the opposite angles and the opposite sides were equal, and concluded that PQRS was a parallelogram (Fig. 1).[3]

(iv) The researcher asked whether they could explain their conclusion without depending on the measured values given on the screen. The students constructed a new quadrangle that appeared to be a rectangle, drew diagonals of PQRS, and measured ∠SOR, ∠SOP, ∠POQ, and ∠QOR. Here, point O is the intersection of two diagonals. They said that if they could show that ΔPOQ≡ΔROS, then they could also conclude that PQ = SR and that the "alternate interior angles" were equal. They searched for a command, which Cabri Geometry does not have, to determine whether the two triangles were congruent.

(v) After the researcher advised the students to use various quadrangles, they dragged the vertices of the initial diagram, which they had used before (iv). They transformed ABCD into a concave quadrangle and noticed that PQRS became a parallelogram even when ABCD was concave. Then they changed ABCD into a quadrangle that appeared to be a rectangle, and into one that appeared to be a trapezoid.

(vi) The researcher advised the students to use the commands available in the Cabri Geometry software. The students used the "polygon" command to draw a square, ABCD. They created PQRS using that square and drew the diagonals of PQRS. After they measured their lengths and measured ∠PSR, ∠PQR, ∠SRQ, and ∠SPQ, they concluded that PQRS was a square. Then they mentioned that PQRS was always a parallelogram.

(vii) When the researcher advised the students to move the diagrams, they dragged the vertices of the initial diagram. They noticed that points P and S did not move even when dragging C. They also noticed that, when dragging B, sides AD and DC did not change and their midpoints were not affected. They mentioned again that PQRS remained a parallelogram. At this point, Yamada asked why the length of PQ did not change when B was dragged.

(viii) The researcher asked why the length of QR did not change even when Q and R were being moved. While Nogawa mentioned that QR was a base side of ΔCQR, Yamada asked again why it did not change. They tried to explain it based on the shape of ΔCQR and checked how its vertices moved when C was dragged. They noticed that PS = QR even when ΔCQR greatly differed from ΔAPS. Yamada asked the same question again and again.

(ix) When asked what did not change when C was dragged, Yamada mentioned the other vertices, A, B, and D. He also mentioned that the way PO changed was the same as that of RO, and that the alternate interior angles were always equal. Then he asked why the length of SO was always the same as that of QO.

(x) The researcher advised the students to use the "replay construction" command, which replays all steps they had taken. When the midpoints were connected to each other to make PQRS in the second replay, Yamada asked why the opposite sides became equal when they merely connected those midpoints. When more than 98 minutes had passed, the researcher intervened to terminate the problem-solving activity.

2. Questions Observed during the Process of Solving Problem 1

Although the students sometimes wondered whether they should call PQRS "a rhombus," they concluded that quadrilateral PQRS became a parallelogram when they constructed the initial quadrangle and measured some of its sides and angles in (i). When vertex D was dragged in (ii), they made the following comments and took for granted that PQRS became a parallelogram.[4]

279 Yamada "What kind of quadrangle does it become? It's easy to

find what kind of quadrangle. Ah, It's easy to find it is a parallelogram, isn't it?"

280 Nogawa "Yes, it's a parallelogram because..."

281 Yamada "The opposite sides are equal."

282 Nogawa "The opposite sides are equal, and the opposite angles are also equal."

When they tried to find the area of PQRS in order to get "crucial evidence" in (iii), they made the following comments.

350 Yamada "It appeared to be a parallelogram, but we cannot conclude so."

However, they immediately concluded that PQRS was a parallelogram after measuring ∠SPR, ∠QPR, ∠SRP, and ∠QRP. At these early stages, the students appeared to have no doubts about their conclusion that PQRS was a parallelogram.

Even when a concave quadrangle appeared on the screen by chance, they voiced no questions.

710 Yamada [seeing the concave quadrangle] "Well, it becomes a parallelogram even in that case."

711 Nogawa "Yeah, somehow it's parallelogram."

712 Yamada "It is a parallelogram wherever it goes."

713 Nogawa "Only angles get..."

714 Yamada "It's a parallelogram even in that case."

715 Nogawa "That's great."

716 Yamada "[inaudible] a parallelogram, isn't it? Well, then let's go back to the main subject."

Although they thought that a concave quadrangle was an exception because they could "not connect its midpoints," they did not question that case and changed the shape of ABCD.

Nogawa explained in (vii) that some midpoints were not influenced by dragging the vertex. After that explanation, Yamada asked why they did not move. This question does not seem focused enough.

856 Nogawa "When moving this point [C], then..."

858 Nogawa "P and Q, no, S and P don't move. Right, S and P don't move."

860 Nogawa "They don't move"

862 Nogawa " When moving this [D], then P and Q"

863 I "P and Q don't move."

864 Nogawa "For now, yeah, I see that."

866 Yamada "Why don't they move?"

867 Nogawa "Because they are midpoints."

869 Nogawa "Because, when moving this B,"

871 Nogawa "the lengths of BA and BC will change,"

873 Nogawa "but the lengths of AD and DC do not change, so their midpoints remain the same."

874 Yamada "Mmm..."

After that, when dragging vertex B on the screen, Yamada asked why the length of PQ did not change.

896 Yamada "Why is only this length [PQ] not changing?"

898 Nogawa "Since the midpoints remain the same."

899 Yamada "So it does not change, even when this becomes very short."

900 Nogawa "That's right."

901 Yamada "It doesn't change."

902 Nogawa "Midpoints, because they are midpoints anyway. Even if the lengths change, the midpoints, well, the midpoints always remain in the middle [of the sides]. The midpoints"

903 Yamada "Then because.... Why? Why does it always become a parallelogram no matter how we move the points?"

904 Nogawa "Because they remain the midpoints."

905 Yamada "That's right. Or because [inaudible] midpoints."

While Yamada asked why the length of PQ did not change and why PQRS became a parallelogram, Nogawa repeated the same reason that the midpoints remained in the middle of the sides even when the vertices were being dragged. Yamada seemed to be convinced by this reasoning at this stage.

When they were dragging vertex C, the researcher highlighted Yamada's question and asked why the length of QR did not change when Q and R were moving. After hearing Nogawa's explanation, Yamada asked by himself why it did not change and attempted to explain it as follows, using the idea of counterbalancing.

956-968 Yamada "Well, when moving C upwards, R moves as if it escapes from [C] and Q moves as if it follows [C]. Conversely, when moving C so that Q moves as if it escapes from [C], then R moves as if it follows [C]. Yes, anyway, the length does not change in such a manner..."

Here, he seemed to explain the fact that the length of QR was constant by referring to the mechanism of the problem situation in which points move in synchronization with each other.(5) However, Yamada immediately asked again why it did not change when dragging vertex C.

After that, they discovered the following.

973 Nogawa "If this [C] is moved, these lengths [CQ and CR] change. Of course, they change. But, this [QR] doesn't change."

The next conversation followed this part.

986 Yamada "[Dragging C] If we know how [CR] decreases, we can find it, can't we?"

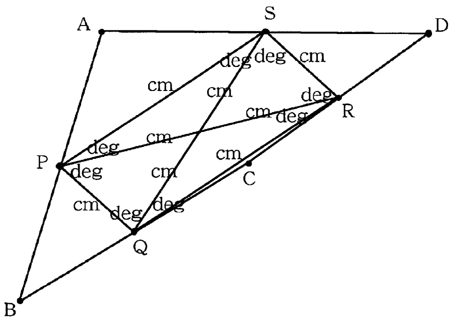

Fig. 2. Diagram in which ΔCQR was crushed ("cm" and "deg" indicate that those sides and angles were measured using the Cabri software, but their exact values could not be read from the VTR data).

987 Nogawa "In some cases, one of them [Q or R] goes up, and in some cases, one of them goes down, and in some cases, both of them go up."

988 Yamada "[Dragging C upward] Well, when moving this vertically, the difference of this and this [CR] gets shortened. So, this [QR] remains the same."

Attending to ΔCRQ, they thought that the sum of lengths of its three sides was constant and attempted to explain the constancy of PQ's length based on this. Indeed, Nogawa said the following after this.

1006 Nogawa "If this [QR] changes, the length of this [CQ] won't change."

Then they noticed that the length of PS was the same as QR even when the sizes of ΔAPS and ΔCRQ greatly differed. This suggested to them that the length of QR was not directly related to the constancy of the sum of ΔCRQ's three sides. After dragging C to QR and squashing ΔCRQ (Fig. 2), Yamada asked the same question.

1048 Yamada "Why does it not change?"

Moreover, he said the following when dragging vertices C, D, A, and B.

1059 Yamada "It's strange that this length does not change no matter how we move it."

After mentioning in (ix) that the length of PO changed in the same way as that of RO, Yamada asked a question about the fact that PO = RO and SO = QO always held.

1104 Yamada "Come to think of it, why do the lengths from them to the intersection not change?"

1105 I "What do you mean by intersection?"

1106 Yamada "Ah, so."

1108 Yamada "The distance from S to the intersection [O]."

1110 Yamada "The distance from Q to the intersection."

1113 Yamada "Why are they always equal?"

1115 Nogawa "Because they are midpoints."

1116 Yamada "I don't agree with that."

Yamada asked the following question during replaying their construction in (x).

1154 Yamada "Why do the opposite sides in Fig. 3 become equal when connecting this midpoint and this midpoint, although the lengths of the sides greatly differ?"

1155 Nogawa "Mmm..."

1156 Yamada "Although the distances to the midpoints also greatly differ, the lengths of the opposite sides become equal when connecting this midpoint and this midpoint."

1157 Nogawa "Mmm..."

When the researcher asked Yamada to repeat his question, he explained it as follows.

1161-1191 Yamada "Ah, well, when taking A, B, C, and D first, we didn't care about the lengths of the sides at all. So, the quadrangle consists of sides whose lengths greatly differ from each other. Nevertheless, when making a midpoint on each side and connecting those midpoints with each other, the opposite sides, the opposite sides or... , yeah the opposite sides become equal. It's strange to me, because line... the distances to the line segments are very different, the distance from A to S is very different [from its correspondent], and the distance from A to P is very different [from its correspondent]."

Yamada mentioned here that quadrilateral ABCD was adopted arbitrarily, and specific conditions were not assumed about the lengths of its four sides. He also mentioned that PQRS was determined only by the condition that the

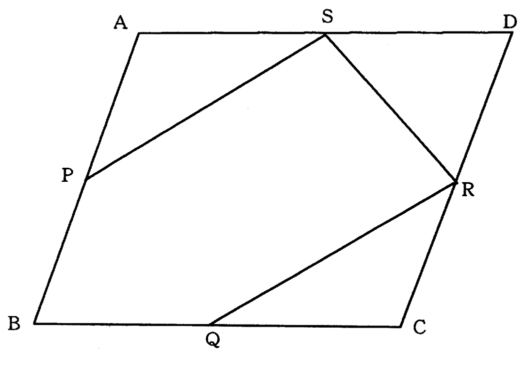

Fig. 3. Diagram which appeared during the replay of the students' construction. Yamada spoke line 1154 when this diagram was on the screen.

midpoints of those sides were connected and no restrictions were given for the lengths of the sides of PQRS. He felt it strange that, although no assumptions were given for the lengths of the sides of ABCD and of PQRS, the opposite sides of PQRS always became equal.

By closely examining their problem-solving processes, we noticed: (1) at later stages, the questions asked had a relation to the mechanisms of the problem situation as their solving process proceeded; and (2) when a solver asked a question relating to those mechanisms, he was not convinced by his partner's superficial explanation, for example "because they are midpoints." Around line 280 in the protocol above, the students paid attention only to the phenomenon of quadrilateral PQRS becoming a parallelogram and they did not question that. Around line 900, Yamada asked why the length of PQ did not change even when dragging B (line 896) and why PQRS remained a parallelogram (line 903). However, such questions were apparently resolved by his partner's explanation, "because they remain the midpoints." Although they noticed that the length of PQ was constant even when AB and BC became shorter during this conversation (lines 899, 901), they seemed to consider these two observations to be phenomena that were simultaneously observed, and they did not think that they were strange.

In lines 956-968, Yamada relates the phenomenon that the length of QR did not change with movement of other elements of the situation (Q and R), or with changes in the other sides of ΔCRQ (CR and CQ). He then found an aspect of the mechanism of the situation, i.e., ΔCRQ can be flat when the length of QR remains the same, and noticed that such movements and changes do not explain that phenomenon. Yamada asked why the length of QR did not change (line 1048) and mentioned that this phenomenon was strange (line 1059) taking into account that aspect of the mechanism. This finding probably suggested to him that the phenomenon was not self-evident. Yamada also asked why SO = QO in 1113, and he was not convinced this time by Nogawa's explanation, "because they are midpoints." When the researcher asked whose location did not change when dragging C, Yamada referred to vertices B, A, and D, while Nogawa referred to midpoints S and P. Moreover, when the researcher mentioned that Q, R, and C were moving, Yamada reacted by saying "yes." It can be said that Yamada was aware of the mechanism of midpoints Q and R moving with C, and asked the question how the constancy of QR related to this mechanism of the situation. He might have thought that since Q and R were moving with C, the distance between Q and R could change when dragging C. Because of the nature of his question, Yamada did not easily accept Nogawa's explanation that Q and R were midpoints.

In asking a question in line 1154, his attention was directed to the relation between the mechanism or the basic structure of the situation and the phenomenon that the opposite sides were equal. Yamada seems to have asked that question because he began to feel a gap between the mechanisms and the phenomenon. Nogawa did not give an explanation like "because they are midpoints" to this question.

IV. Questions Arising while Solving Problem 2

1. Outline of Their Solution to Problem 2

(i) To construct an equilateral triangle, BAD, they drew a perpendicular bisector of AB using one of the Cabri commands. Nogawa operated the software (which he did throughout the problem-solving process). Then they took a point D on the line and moved it while visually checking whether $\angle$ABD was 60°. They constructed ΔACE and ΔBCF in the same way and made a quadrilateral ADFE. They said that ADFE appeared to be a parallelogram and measured its four interior angles. They also measured the lengths of AE and EF, and found that their values differed from the lengths of AD and DF, which they had measured before. They thought that this was strange.

(ii) Since they had visually constructed the equilateral triangles, the shapes of the figures became distorted during measuring the sides and angles. Although they tried to make new constructions, they always did this visually. The researcher advised them to use a computer file which he had made in advance. In this file, the problem situation was constructed using the appropriate Cabri commands, so that the given conditions were maintained regardless of how the original triangle ABC was transformed.

(iii) The students opened that file. They found that the figures moved hand-in-hand when vertex A was dragged, and the four angles and four sides of quadrilateral ADFE were measured (Fig. 4). When dragging A, they said that ADFE was a parallelogram. They continued dragging and overlapped A on BC and on F (Fig. 5). In doing so, Yamada asked why ADFE became a parallelogram. Using the "parallel?" command, they checked whether AE and AD were parallel to DF and EF respectively. Nogawa concluded that ADFE was a parallelogram.

(iv) They thought that ADFE was a parallelogram because the opposite sides were equal and the opposite angles were also equal. When the researcher asked why they remained equal when the figures

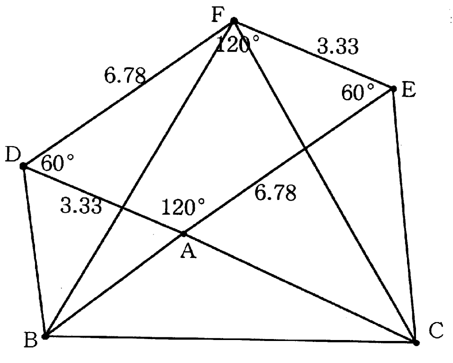

Fig. 4. Diagram which appeared when the file was opened. The sides and angles were measured by the students.

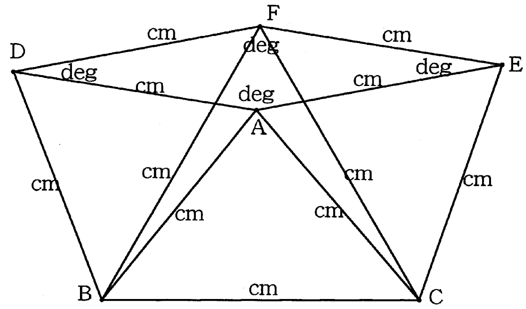

Fig. 6. Diagram in which students made ΔABC an isosceles triangle.

were moved, the students began to explore the situation again. In dragging A and C, Yamada asked why the opposite sides became parallel to each other.

(v) When the researcher advised them to use other functions of the Cabri software, they adopted the "replay construction" command without hesitation. When vertices D and F were connected in the replay, Yamada asked why the opposite sides became parallel to each other. Following the researcher's advice to measure something, they measured the lengths of CE, AC, AB, BD, and BC. Although they also measured ∠DBF and ∠FBA, they immediately deleted their values on the screen and measured the lengths of FB and FC.

(vi) When the researcher asked what did not change when dragging the figures, they moved sides AC and AB, and Nogawa said that ΔABD did not change when AB was dragged. Yamada said when dragging A, triangles without dragged points did not change. After the researcher advised them to make ΔABC a special triangle, they made it to be equilateral and then isosceles (Fig. 6). Nogawa found that the lengths of all sides except ΔFBC's sides were the same, at 8.59. Yamada asked why the length of DF was also 8.59. He found that ΔBFD and ΔECF were also isosceles triangles and asked why they had become that way.

(vii) After dragging A and C, Yamada mentioned that ΔBFD and ΔECF were isosceles if ΔABC was isosceles, and that ΔBFD and ΔECF were not isosceles if ΔABC was not isosceles. They dragged vertices and then made ΔABC an isosceles triangle again. Yamada mentioned that the lengths of BF and BD were known because ΔBAD and ΔBCF were equilateral, and he insisted that, because of these lengths, the length of DF could be determined automatically. They checked whether EF's length could be also determined and understood that opposite sides became equal if ΔABC was isosceles.

(viii)To examine the case of non-isosceles triangles, they moved vertex A slightly. Seeing this, Nogawa noticed that ΔBFD and ΔECF were the size of ΔABC, and Yamada said that ΔABC could be placed on ΔBFD if it was rotated around B and it could be placed on ΔECF if it was rotated around C. Based on this rotation, they could demonstrate that EF = AB, which implied that AD = EF and that the opposite sides were equal. Although they demonstrated this, they still depended upon the

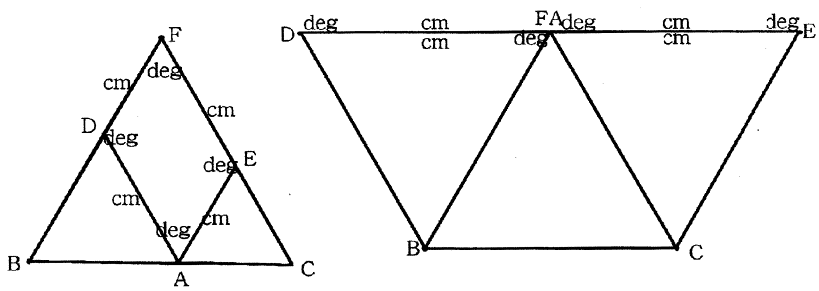

Fig. 5. Diagrams which appeared during students' dragging of point A in Fig. 4.

"parallel?" command and measuring the values of some angles. Since more than 110 minutes had passed, the session was ended by the researcher.

2. Questions Observed in the Process of Solving Problem 2

Similar to their process for solving problem 1, they said that ADFE was a parallelogram as soon as they had completed its construction and saw ADFE on the screen.

160 Nogawa "What kind of quadrangle does it become?"
161 Yamada "It appears to be a parallelogram."

Even when the measured values were inconsistent with their conclusion, they did not abandon it and blamed their use of the software, such as when they said, "Did we make a mistake?" and "There is something wrong" (cf. Nunokawa (1997)). Here, they did not question the phenomenon of ADFE becoming a parallelogram.

In (iii), Nogawa asked a question when they used the file the researcher had prepared in advance.

528 Yamada "All the figures move hand-in-hand when moving it."
529 Nogawa "Why?"
530 Yamada "I have no idea."
531 Nogawa "Yeah, all of it is moving. It's wonderful. So, what should we do?"

Since their constructions were not based on the appropriate commands, figures could not move and maintain the given conditions. Nogawa seems to have questioned the nature of the construction, not why ADFE had become a parallelogram.

When dragging A and putting it on side BC in (iii), Yamada, when seeing this special case, asked why ADFE became a parallelogram.

552 Yamada "That's not good."
553 Nogawa "Ha ha... But, it is at least ADFE, a quadrangle. Anyway, this is also OK, isn't it?"
554 Nogawa "That doesn't happen by chance, does it?"
555 Yamada "Something is hidden. [Dragging A]"
563 Nogawa "[A diagram like Fig. 4 is on the screen] Like this. This is a parallelogram again."
574 Yamada "[A lies on BC] It becomes a parallelogram no matter where we move it."
582 Nogawa "[A, which overlaps F, is dragged toward BC] Its shape is a parallelogram."
583 Yamada "Why do we get a parallelogram?"
588 Nogawa "Why... Look, look! This is a mysterious phenomenon. Look, this shape is beautiful, isn't it? [Moving A from BC and putting it upon F]"
589 Yamada "A parallelogram..."
590 Nogawa "You're right. That's a parallelogram."
591 Yamada "Why a parallelogram?"
592 Nogawa "What? Because its opposite sides are equal."
593 Yamada "That's right."

Here, they thought that a quadrangle in a special case did not become a parallelogram by chance and that something hidden underlay this phenomenon. In addition, Yamada asked why ADFE became a parallelogram. However, he asked about ADFE becoming a parallelogram without relating it to the mechanism of the problem situation. When Nogawa gave the reason that the opposite sides were equal, he accepted it. Immediately after this, Yamada proposed using the "parallel?" command and when it showed that the opposite sides were parallel, he agreed with the conclusion that ADFE was a parallelogram saying, "That's enough." These activities are consistent with the nature of his question.

Following the researcher's intervention in (iv), Yamada asked why the opposite sides were equal, why the opposite angles were equal, and why ADFE was a parallelogram. Furthermore, when Nogawa was dragging vertex A, Yamada asked the following question.

644 Yamada "Why do D and E also move when A is moved?"
645 Nogawa "What?"
646 Yamada "That's a question."
647 Nogawa "Because they are vertices of equilateral triangles. As D and E are vertices of the equilateral triangles, AEC and ADB ..."
648 Yamada "[inaudible] Then, why do those opposite sides become parallel?"
649 Nogawa "If moving this [C], then D and A are..."
650 Yamada "That's mysterious."

Yamada focused on the movement of D and E, and he seems to have asked why ADFE became a parallelogram in relation to their movement. He might have thought that, although D and E were moving, AD was always parallel to EF and DF was always parallel to AE. The term "mysterious" he used suggests that he thought this phenomenon was not self-evident.

When Nogawa dragged C and A widely, Yamada asked a similar question.

662 Yamada "Why on the earth does this become a parallelogram? We take equilateral triangles of different sizes. Although we take such equilateral triangles, why do the opposite sides become equal to each other?"
674 Yamada " I know that it's a parallelogram. How does it become a parallelogram?"
675 Nogawa "That's a question. Everything will be OK if we know that."

676 Yamada "Anyway, why does it become a parallelogram?"

Yamada related the phenomenon of ADFE becoming a parallelogram to a part of the mechanism of the problem situation. That is, he might have thought that, although the equilateral triangles could be very different sizes and the locations of D, E, and F could vary rather freely, the opposite sides including these points were always parallel to each other. Nogawa gave no reason this time.

While replaying the constructions in (v), Yamada asked about the relationship of the opposite sides becoming parallel to each other.

779 Yamada "Why do the opposite sides already become parallel when drawing this line [DF]?"

803 Nogawa "[Going back to the point where the three equilateral triangles had been drawn] All we can do here is to connect the points by lines."

804 Yamada "[Going back to the earlier steps, when F was taken in the replay] I doubt the steps from about here."

They seem to have related their questions to a certain aspect of the mechanism of the situation, i.e., the opposite sides only connected the points that were pre-determined by the equilateral triangles. Before asking this question, Yamada mentioned another aspect of the situation in this replay: "These points [A, B, and C] are taken almost arbitrarily" (line 767). He was aware that the locations of A, B, and C had no specific features. This suggested that he might have asked the above question in relation to the way of taking the vertices of the original triangle ABC.

After asking this question, Nogawa mentioned that BD = AB and the length of FB was known without measuring it. They paid attention to the lengths that were known according to the given equilateral triangles. This illuminated the lengths that were not obviously known from the given conditions.

When dragging sides AC and AB in (vi), Nogawa mentioned that ΔACE and ΔABD did not change, respectively. When dragging point A, Yamada said that the triangle with no moving point did not change. These observations concern the dependence of the lengths on other components. After they made ΔABC isosceles and found that many sides became 8.59 units long, Yamada asked a question about some lengths following the researcher's intervention.

947 Yamada "I know why these two lines [AD and AE] became 8.59 long. But I don't know why the two lines above that also become 8.59 long."

955-965 Yamada "Well, ABC, no, triangle ABD, if all the sides of that triangle become 8.59 long, then this [AD] becomes 8.59 long. This [AE] also becomes 8.59 long similarly. But, although the two lines above that are merely lines connecting..."

Here, the question about the lengths of DF and EF was related to an aspect of the mechanism of the situation: DF and EF were not directly dependent upon the given equilateral triangles. Such a mechanism clarified that some relations among the lengths were not obvious. After mentioning this, Yamada noticed that ΔEFC and ΔDBF were isosceles when ΔABC was isosceles. He did not merely observe this phenomenon, but also asked why all of them became isosceles triangles (line 981). One can say that this question is also supported by the same point that DF and EF are not necessarily equal to the sides of the equilateral triangles. This question led them to notice in (viii) that those three triangles were congruent, and that this congruence could explain why ADFE became a parallelogram.

Similar to their process for solving problem 1, the students asked no questions about the phenomenon of ADFE becoming a parallelogram in the early stages. Although, seeing the special case on the screen, Yamada asked why it became a parallelogram in lines 583 and 591 in the protocol, he seems to have been convinced by Nogawa's explanation that the opposite sides were equal. When Yamada asked why the opposite sides were equal relating this question to movement of D and E in line 648, he felt that this phenomenon was "mysterious." In lines 662 and 674, Yamada related the questions to the mechanism of the problem situation, i.e., the situation consisted of the three equilateral triangles of different sizes, and vertices D, E, and F could vary rather freely. In this case, Nogawa avoided giving a simple reason for it. After trying to relate their questions with some parts of the situation in line 804 and paying attention to the dependence of the length of the sides, Yamada asked about the length of DF and EF while taking account of the mechanism of the situation. Nogawa joined his activity to investigate triangles congruent to ΔABC.

Their questions gradually became related to their understanding of the problem situation, after which, the questions were not resolved by superficial reasons.

V. Questions and Understanding of Problem Situations

1. Limited Understanding of Mathematical Proofs

As shown in Section III.1 and Section IV.1, in earlier stages of their problem-solving process, the students supported their conclusions using the results of Cabri functions, i.e., by measuring sides and angles and checking parallel relationships. While the Cabri settings might facilitate their tendency to use these measurements, it seemed difficult for them to pursue mathematical proofs even when they tried to think in a paper-and-pencil manner. In stage (iv) of the first problem-solving session, they were asked to explain their conclusion if it did not depend on the Cabri functions. Although they tried to tackle the problem as if they were in a paper-and-pencil setting, they mentioned measurement with rulers and they wanted to measure the angles. In the final part of the second session, the students also mentioned measurement using protractors in spite of trying to think in a paper-and-pencil manner.

Kumakura (1999) investigated whether Japanese eighth and ninth graders appreciated the role and significance of mathematical proofs. His research showed that more than 70% of ninth graders thought that experiments and measurements were sufficient to support a geometrical statement. Some activities of Yamada and Nogawa displayed a limited understanding of mathematical proofs that is similar to that of ninth graders reported by Kumakura (1999). Their "emotional orientation to mathematics" (Drodge & Reid, 2000) seemed to differ from ours. Furthermore, their activities also remind us of the second level of van Hiele's theory (Nunokawa, 1992). To verify their conclusions, they tried to gather evidence for their conclusions (*e.g*., by measuring four sides, measuring four angles, checking whether the opposite sides were parallel, etc.), rather than trying to prove those conclusions. In such a case, the teacher and students may be speaking different languages (van Hiele, 1986: p. 90). The fact that the students merely listed properties of PQRS when asked why it had become a parallelogram implies that the question the researcher asked did not mean what he had intended to the students. The students seemed satisfied with their conclusions which were supported only by measurements and the result of the "parallel?" command in the earlier stages of the problem-solving process. Even when Yamada spontaneously asked a "why" question in solving problem 2 (line 583), he was satisfied with Nogawa's comment which was based on the measurements. Thus the researcher sometimes needed to intervene and ask some questions. Since the students were influenced by the researcher to begin to ask "why" questions, it is difficult to draw conclusions about the origins of the "why" questions. However, it is possible to observe some changes in how they reacted to the "why" questions. Although, as mentioned above, the researcher's "why" questions did not seem to make sense to them in the beginning, the students appropriated those questions and searched for reasons that would explain their conclusions in the later stages. In this sense, "why" questions became genuine ones at the later stages of their problem-solving process. Such changes in their reaction to "why" questions were supported by their understanding of the problem situation.

2. Importance of Understanding the Problem Situations

As shown in the previous sections, the students, as their solving processes progressed, could gradually relate their questions to mechanisms of the problem situations. For example, Yamada asked why the length of DF was equal to the lengths of AE and AC in stage (vi) of the second session. In this case, he did not merely ask this question, but also associated this question with the mechanism of the problem situation, i.e., that the length of DF was determined by vertices D and F whose locations did not seem to directly depend upon the length of AC. When they related their questions to some aspects of the problem situations, the students began to adhere to those questions and were not satisfied with reasons based only upon measurements and the results of Cabri functions. They seemed to want to explain their conclusions using mechanisms of the problem situation. That is, a "why" question became a genuine question for them, and the explanations they could accept seemed to change as the problem-solving process proceeded.

In the first session, Yamada felt it was strange that the length of QR did not change regardless of how they dragged vertex C (line 1059). He also felt it mysterious in line 650 of the second session that the opposite sides became parallel regardless of how vertex A was moved. It can be said that he recognized a discrepancy in two aspects of the problem situation (*e.g*., the constancy of the length of QR and the motion of vertex C). Nunokawa (2001) said that, in order to feel surprises or gaps, it is necessary for learners to have some expectations about the object at issue. And, in order to have expectations about that, subjects are required to understand the object to some extent. In fact, Yamada's feeling about the length of QR was supported by his understanding of the problem situation. Before manifesting this feeling, he found that, when dragging C, points Q and R also moved, and that ΔCRQ could be crushed and then the lengths of CQ and CR changed rather freely. He also noticed that, even when ΔCRQ greatly differed from ΔAPS, QR became equal to SP. Such understanding of the situation elicited his recognition of the gap and supported his having a genuine "why" question.

Some researchers have proposed a concept of understanding as the establishment of a network of ideas and facts (*e.g*., Hiebert & Carpenter (1992)). Taking this standpoint, it can be said that questions themselves make good sense when they are incorporated into a network of ideas and facts about a problem situation. Such incorporation may help solvers ask questions about a question or the meta-question: "Why should I ask such a question?" Exploring and understanding problem situations are important not only for resolving questions in problems (cf.

Nunokawa (2000)), but also for making sense of those questions. From this viewpoint, it is natural that the students could ask genuine "why" questions only after they had explored the problem situations and gradually understood them. Their exploration and understanding of the problem situations might have been an important factor supporting their "why" questions.

At the end of the second session, the students could not finish their proof since they could not show appropriate reasons why ΔABC could be put on ΔBFD and ΔECF by rotation. If they had noticed that such a phenomenon was not obvious and felt that it was worth explaining, they would have pursued a proof based more on mathematics.

3. Roles of Dynamic Geometry Software

It is obvious that the students' activities with dynamic geometry software facilitated their exploration and understanding of the problem situations which were important in their developing "why" questions. Dragging made it possible for the students to observe mechanisms of the problem situations. By replaying the constructions, they could examine how a certain element was determined by other elements. While the measurement function of the Cabri software might have elicited the students' explanations based on measurements, it also facilitated their recognition of dependent relationships among the elements in the problem situations. If the lengths of DE and AC in Fig. 6 are displayed on the screen and vertex A is dragged, it might be easy to notice that the length which DE changes depends on that of AC.

Balacheff & Kaput (1996: p. 476) said that in dynamic geometry settings, the statement of a geometrical property becomes a description of a geometrical phenomenon accessible to observation and that these phenomena occur in fields of experimentation. Other researchers have pointed out that dynamic geometry software facilitates making geometrical properties of figures explicit (Mariotti & Bartolini-Bussi, 1998; Tsuji, 1997). In the first and second sessions, the conclusions that the target quadrangles, PQRS and ADFE, became parallelograms appeared as observable phenomena on the screen. What became observable was, however, not only the phenomena directly concerning the conclusions, but also the mechanisms of the problem situations. The Cabri software seemed to make it possible to operate upon or "touch" the problem situations. The students could alter the situation by dragging the figure and check their results with measurements on the screen. Such exploration also highlighted some discrepancies in the observed properties and those mechanisms and facilitated the students asking genuine "why" questions.

As discussed in Section V.1, the students were apt to be satisfied with their conclusions that were supported only by measurements and the result of the "parallel?" command. The data of this study suggest that some students need teacher support to direct their attention to mechanisms that can fully explain the target phenomena. Such support would help students to use the software effectively to develop mathematical proofs. When beginning to draw a new diagram in the first session, Yamada said that too many measurement values made him confused (lines 470-472), and that they should measure only necessary parts (line 509). Moreover, he did not operate the software during their problem-solving process. In this sense, he seemed to be less dependent upon the software. However, he also seemed to relate his "why" questions to his understanding of the problem situations more easily. The influence of the dependence of solvers on software needs to be investigated in future research.

VI. Concluding Remarks

In the introduction of this paper, we mentioned an emotional aspect on the side of solvers. The above analysis suggests that the students' feeling of "why?" was influenced by their understanding of a problem situation. Even if mathematics teachers attend to the importance of "why" questions and ask our students why a certain statement is true, it may be the case that our students are not interested in searching for proofs that explain the statement. In order for "why" questions to become genuine questions for students, opportunities to explore and/or understand problem situations should be provided.

The students in this study had stronger feelings of "why?" in the later stages of their problem-solving process. In the second session, such feelings led them to explain their conclusions using the three congruent triangles in the problem situation. When he found these triangles, Nogawa said that he had made "a big discovery." The students might have appreciated "proofs that explain" and have experienced geometry problem solving in the sense that they searched for reasons of their "why?". Before they could have feelings of "why?", however, the students were prompted by the researcher to ask "why" questions several times. Factors that stimulate solvers to spontaneously ask "why" questions need to be explored in future research.

Notes

(1) Using the Cabri Geometry software, solvers can construct basic figures (*e.g.*, points, lines, circles, parallel lines, and perpendicular lines) and their combinations. This means that solvers can construct a rectangle in the same way as with the usual ruler-compass construction. Solvers can also measure constructed elements (*e.g.*, lengths, angles, and areas) and move points and lines by dragging

with given conditions satisfied (*e.g.*, parallel).

(2) Measured values remain near the measured components. For example, the length of a side continues to be shown near that side. Its value can change in accordance with changes of the sides.

(3) The Cabri software shows the values of lengths only using its numerical values without units (*e.g.*, cm), while it shows the measures of angles using its numerical values with the unit of degrees, "∘".

(4) The number at the head of each utterance corresponds to the number of the line in the protocol in Fukuzawa (2001).

(5) The term "mechanism of the problem situation" is used here to highlight the dependence relationships in the problem situation. For example, when moving vertex C in Fig. 1, vertex R also moves depending upon the motion of C. As shown in Fig. 2, ΔCQR can be crushed independent of the length of QR. Such dependence (or independence) is referred to here.

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學生在數學解題過程中問題的產生及對問題的瞭解

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摘 要

過去的研究指出解題者在數學解題過程中提問「爲甚麼？」是十分重要的，本文分析詢問「爲甚麼？」的本質，以增加對這類問題的瞭解。爲達此目的，本文分析了兩位九年級學生用動態幾何軟體解兩道題時的表現，他們或自發地、或在引導下數度提問了「爲甚麼？」。分析顯示當學生探索問題的情境及深入瞭解這些情境的時候，他們會對操作幾何軟體時所產生的情境提問「爲甚麼？」進而會認眞探討。這結果表示解題者能夠眞實發問「爲甚麼？」是建基於他們對問題情境的瞭解，這也表示若要學生能夠提出眞誠的疑問，則讓他們有機會探索問題情境到若干程度是十分重要的。