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Free Vibration Analysis of General Plates Using a New Element

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ABSTRACT

The free vibration analysis of general plates using a newly developed four-noded conforming plate element is presented. The plate formulation is based on the classical Kirchhoff thin plate theory and energy orthogonality theorem and can automatically satisfy the convergence requirements. It employs bilinear functions to perform two-dimensional geometric interpolation. The displacement functions are those of modified bicubic polynomials which contain a rigid body, constant strain and higher order deformation modes. The four-noded element has 16 degrees of freedom in total. The displacement functions and degrees of freedom are expressed and investigated in Cartesian form. The numerical results of the new element are compared with the available data in the literature based on the effects of element geometry and the irregularity of mesh.

Key Words: plate finite element, free vibration analysis, energy orthogonality theorem, transverse displacement functions

I. Introduction

Engineering problems which involve the ability to predict the free vibration behavior of plate structures may generally be solved by using the finite element method. However, as there are a variety of theoretical and numerical formulations for the method, many plate bending elements, each having some merits and demerits, have been developed over the years (Hrabok and Hrudey, 1984; Batoz *et al.*, 1980). To date there is still no agreement among researchers on the *best* or *optimum* element available.

In the finite element method, the structure is represented by an assembly of elements, each having an appropriate stiffness. Numerical finite element calculations are based directly on the element stiffness matrices. It may, thus, be said that the characteristics of a finite element are hidden among the actual numbers in its stiffness matrix.

In the bending of thin plates, it is impossible to construct a strictly C^1 polynomial interpolation for general three-node triangular and four-node quadrilateral elements with three kinematic degrees of freedom (DOF) per node (Batoz *et al.*, 1980; Batoz and Tahar, 1982). Conditions of interelement compatibility are violated by such a displacement field. Continuity of deflections at the interface is the prime requirement which must be satisfied. The next requirement is the continuity of slopes which, if achieved, would produce a continuous, piecewise differential field of displacements in the structures. Success in achieving full conformity comes easiest in the case of rectangular elements. Bogner *et al.* (1966) developed 16 and 36 DOF conforming rectangles that exhibited good convergence properties. It was necessary, however, to use second derivatives of displacement as DOF. In particular, the 16 DOF element the twist was used.

To achieve a conforming element, while still using only the three geometric DOF, several researchers (Bazeley *et al.*, 1965; Gallagher, 1969; Clough and Tocher, 1965) have used different approaches to meet the requirements. Of all the different schemes considered for derviation of conforming plate bending elements, perhaps the most straightforward is the use of higher order polynomials. A comprehensive review of the different plate bending elements based on their variational formulation can be found in Hrabok and Hrudey (1984).

The plate bending elements are normally derived from the potential energy or a hybrid formulation. A radically different and promising approach is the direct method introduced by Bergan and Hanssen (1976). This method is not based on a variational principle. Instead, the elements are derived directly from the conditions that they satisfy the patch test and the rigid body motion and constant strain re-quirements, and a variety of such C^0 elements have been tested (Bergan, 1980; Bergan and Nygård, 1981). Although successful applications have been shown in many cases, little or no emphasis has been given to the behavior of these elements in C^1 continuity.

This paper presents a formulation of a four-noded sixteen-DOF. quadrilateral thin plate bending element in terms of the Cartesian co-ordinates, which is based on the so called "free formulation" as formulated by Bergan (1980) and Liu and Lin (1993). The formulation presented originally by Liu and Lin (1993) in a somewhat different form uses a slightly modified bicubic polynomial in the natural coordinates ξ and η as the displacement function for w. These displacement patterns contain the fundamental deformation modes and the higher order displacement modes, and perform in a linearly independent manner. The use of Cartesian coordinates, which appears to be the natural choice for quadrilateral elements, makes it possible to obtain conveniently the explicit expressions for the unknown coefficients in the displacement functions in terms of the nodal displacements; thus, the time consuming numerical procedure is avoided. Also, transformation of partial derivatives from the natural coordinate system to the rectangular coordinate frame is rather cumbersome. This involves inversion of Jacobian matrices for all levels of differentiation. It should be noted that the usual approach of expressing transverse displacements and rotations by separate expansions is not allowed here.

The objectives in this paper are to show how the general continuum mechanics based plate element formulation of Liu and Lin (1993) can be applied to free vibration analysis of plate structures, and to give some insight into the plate element formulation.

II. Formulation of the New Element

The formulation closely resembles that of the NCQ (Liu and Lin, 1993) element. It starts with the free formulation (Bergan and Hanssen, 1976; Bergan, 1980), where the transverse displacement w of the element is described by way of generalized displacement patterns or "modes":

$$w=\mathbf{N}_a\mathbf{a}=\mathbf{N}_{rc}\mathbf{a}_{rc}+\mathbf{N}_h\mathbf{a}_h, \tag{1}$$

where $\mathbf{N}_{rc}$ expresses a complete polynomial to a degree which corresponds to a complete rigid-body and constant straining field expansion (rc-modes) for the plate, and $\mathbf{a}_{rc}$ are the associated polynomial coefficients. N_h expresses a set of higher order deformation functions (h-modes) and $\mathbf{a}_h$ are the associated coefficients. The total number of rc-modes plus h-modes must be equal to the number of degrees of freedom in the nodal displacement vector $\mathbf{q}$ for the element.

The quadrilateral element has four corner nodes, as shown in Fig. 1, each of which has four degrees of freedom: w; $\partial w/\partial x$; $\partial w/\partial y$; $\partial w^2/\partial x\partial y$. Thus, the vector $\mathbf{q}$ is

$$\mathbf{q}=\begin{Bmatrix}\mathbf{q}_1\\\mathbf{q}_2\\\mathbf{q}_3\\\mathbf{q}_4\end{Bmatrix},\quad \mathbf{q}_j=\begin{Bmatrix} w\\ \dfrac{\partial w}{\partial x}\\ \dfrac{\partial w}{\partial y}\\ \dfrac{\partial^2 w}{\partial x\partial y}\end{Bmatrix}_j \quad ;\ j=1,\ 2,\ 3,\ 4. \tag{2}$$

The kinematic relationship between the nodal degrees of freedom $\mathbf{q}$ and the generalized modes in Eq. (1) is easily obtained by inserting the appropriate nodal co-ordinates into $\mathbf{N}_a$:

$$\mathbf{q}=\mathbf{G}\mathbf{a}=\mathbf{G}_{rc}\mathbf{a}_{rc}+\mathbf{G}_h\mathbf{a}_h, \tag{3}$$

where $\mathbf{G}$ is quadratic, and $\mathbf{N}_a$ must be chosen so that it is invertible. Vector $\mathbf{a}$ is then established by Eqs. (1) and (3):

$$\mathbf{a}=\begin{Bmatrix}\mathbf{a}_{rc}\\\mathbf{a}_h\end{Bmatrix}=\mathbf{G}^{-1}\mathbf{q}=\mathbf{H}\mathbf{q}=\begin{bmatrix}\mathbf{H}_{rc}\\\mathbf{H}_h\end{bmatrix}\mathbf{q}\,. \tag{4}$$

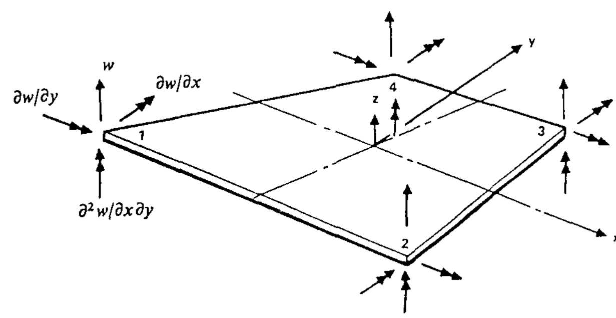

Fig. 1. Geometry and degrees-of-freedom for the four-node quadrilateral plate bending element.

Substitution of Eq. (4) into Eq. (1) yields

$$w=\mathbf{N}_{rc}\mathbf{H}_{rc}\mathbf{q}+\mathbf{N}_h\mathbf{H}_h\mathbf{q}=\mathbf{N}\mathbf{q}; \quad \mathbf{(5)}$$

thus,

$$\mathbf{N}=\mathbf{N}_{rc}\mathbf{H}_{rc}+\mathbf{N}_h\mathbf{H}_h. \quad \mathbf{(6)}$$

The computation of strains from Eq. (5) is straightforward:

$$\epsilon=(\Delta\mathbf{N}_{rc}\mathbf{H}_{rc}+\Delta\mathbf{N}_h\mathbf{H}_h)\mathbf{q}=(\mathbf{B}_{rc}\mathbf{H}_{rc}+\mathbf{B}_h\mathbf{H}_h)\mathbf{q}=\mathbf{B}\mathbf{q}, \quad \mathbf{(7)}$$

where Δ is the appropriate strain-producing differential operator. Based on the displacement expansion on generalized functions and the strains in Eq. (7), the potential energy of the plate is

$$U=\frac{1}{2}\mathbf{q}^T\int_V \mathbf{B}^T\mathbf{C}\mathbf{B}\,dV\mathbf{q}=\frac{1}{2}\mathbf{q}^T\mathbf{K}\mathbf{q}, \quad \mathbf{(8)}$$

where V is the volume of the element, and $\mathbf{C}$ is the constitutive matrix of an isotropic plate.

The generalized stiffness matrix may be partitioned according to rc-modes and h-modes:

$$\mathbf{K}=\begin{bmatrix}\mathbf{K}_{rc} & \mathbf{K}_{rch}\\ \mathbf{K}_{hrc} & \mathbf{K}_h\end{bmatrix}, \quad \mathbf{(9)}$$

where

$$\mathbf{K}_{rc}=\int_V (\mathbf{B}_{rc}\mathbf{H}_{rc})^T\mathbf{C}(\mathbf{B}_{rc}\mathbf{H}_{rc})\,dV$$

$$=V(\mathbf{B}_{rc}\mathbf{H}_{rc})^T C(\mathbf{B}_{rc}\mathbf{H}_{rc}) \quad \mathbf{(10a)}$$

$$\mathbf{K}_{rch}=\int_V (\mathbf{B}_{rc}\mathbf{H}_{rc})^T\mathbf{C}(\mathbf{B}_h\mathbf{H}_h)\,dV \quad \mathbf{(10b)}$$

$$\mathbf{K}_{hrc}=\int_V (\mathbf{B}_h\mathbf{H}_h)^T C(\mathbf{B}_{rc}\mathbf{H}_{rc})\,dV \quad \mathbf{(10c)}$$

$$\mathbf{K}_h=\int_V (\mathbf{B}_h\mathbf{H}_h)^T\mathbf{C}(\mathbf{B}_h\mathbf{H}_h)\,dV. \quad \mathbf{(10d)}$$

When a finite element is derived from a complete set of constant strain and rigid body modes plus a set of linearly independent higher order modes which is energy orthogonal to the first set, the element is convergent (Bergan and Hanssen, 1976; Bergan, 1980). The requirement of orthogonality between the rc-modes and the h-modes is that $\mathbf{K}_{rch}=\mathbf{K}_{hrc}=0$, which implies that there should be no coupling in energy between them. Since the rc-modes produce a $\mathbf{B}_{rc}$ which is constant over the volume (see Eq. (10a), $\mathbf{K}_{rch}=\mathbf{K}_{hrc}=0$ may be satisfied by requiring

$$\int_V \mathbf{B}_h dV=0. \quad \mathbf{(11)}$$

It is evident that the element stiffness matrix may be constructed as

$$\mathbf{K}=\mathbf{K}_{rc}+\mathbf{K}_h. \quad \mathbf{(12)}$$

The simplest selection of the transverse displacement w in the element is assumed to consist of a 16 terms of bicubic polynomials in the x–y coordinates:

$$w=a_1+a_2x+a_3y+a_4x^2+a_5xy+a_6y^2+a_7x^3+a_8x^2y$$

$$+a_9xy^2+a_{10}y^3+a_{11}x^3y+a_{12}x^2y^2+a_{13}xy^3+a_{14}x^3y^2$$

$$+a_{15}x^2y^3+a_{16}x^3y^3=\mathbf{N}_a\mathbf{a}=\mathbf{N}_{rc}\mathbf{a}_{rc}+\mathbf{N}_h\mathbf{a}_h, \quad \mathbf{(13)}$$

in which

$$\mathbf{N}_{rc}=[1\ x\ y\ x^2\ xy\ y^2] \quad \mathbf{(14)}$$

and

$$\mathbf{N}_h=[x^3\ x^2y\ xy^2\ y^3\ x^3y\ x^2y^2\ xy^3\ x^3y^2\ x^2y^3\ x^3y^3]. \quad \mathbf{(15)}$$

A major disadvantage of the choice of the higher order deformation modes in Eq. (15) is that it contains three fourth degree terms and one sixth degree term, and they depend on the reference frame ($\mathbf{G}$ matrix may become singular for certain element shapes). The functions in Eq. (15) must, therefore, be discarded for general quadrilaterals.

An alternative to the modes in Eq. (15) is

$$\mathbf{N}_h=[x^3\ x^2y\ xy^2\ y^3\ x^3y-\alpha_1xy\ x^2y^2-(\alpha_2x^2+\alpha_3y^2)$$

$$xy^3-\alpha_4xy\ x^3y^2\ x^2y^3\ x^3y^3-\alpha_5xy]. \quad \mathbf{(16)}$$

These h-modes are energy orthogonal. They also result in a positive eigenvalue. The coefficients α_i (i=1, 2, 3, 4, 5) should be chosen such that the corresponding terms satisfy the orthogonality condition (11); thus,

$$\int_A (3x^2 - \alpha_1)\, dA = 0 \tag{17a}$$

$$\int_A (y^2 - \alpha_2)\, dA = 0 \tag{17b}$$

$$\int_A (x^2 - \alpha_3)\, dA = 0 \tag{17c}$$

$$\int_A (3y^2 - \alpha_4)\, dA = 0 \tag{17d}$$

$$\int_A (9x^2y^2 - \alpha_5)\, dA = 0\,, \tag{17e}$$

where α_i are easily determined from Eqs. (17a)-(17e), and A is the area of the element. It is recommended that a translated and scaled local x–y system to the centroid of the element area be used rather than the global system. The function of Eq. (16) does not lead to any difficulties in connection with inversion of **G**.

III. Element Stiffness Matrix

The main objective of a finite element development in stress analysis is to derive the element stiffness matrix. This stiffness matrix must satisfy the patch test or, equivalently, the individual element test (Bergan and Hanssen, 1976; Bergan, 1980) in order to give reliable results. The basis for the individual element test is that the element should, when deformed in an rc-state, generate nodal forces (moments) which are the same as the forces obtained by a consistent lumping of the edge tractions to the nodes. In mathematical terms, this may be written as

$$\mathbf{T}_{rc}=\mathbf{P}_{rc}\mathbf{a}_{rc}, \tag{18}$$

where $\mathbf{P}_{rc}$ expresses the nodal forces produced by the generalized rc-modes. $\mathbf{T}_{rc}$ is the nodal forces produced by lumping of tractions at the element boundary to the nodes. In practice, it is possible to factorize $\mathbf{P}_{rc}$ in the following manner:

$$\mathbf{P}_{rc}=\mathbf{LCB}_{rc}, \tag{19}$$

where **L** is the lumping matrix which depends on the element geometry.

The nodal forces $\mathbf{S}_{rc}$ generated by the element stiffness matrix **K** during the rc-state are

$$\mathbf{S}_{rc}=\mathbf{K}\mathbf{q}_{rc}=\mathbf{K}\mathbf{G}_{rc}\mathbf{a}_{rc}. \tag{20}$$

The individual element test demands that the forces $\mathbf{S}_{rc}$ in Eq. (20) be equal to $\mathbf{T}_{rc}$ of Eq. (18) for all $\mathbf{a}_{rc}$. Thus,

$$\mathbf{K}\mathbf{G}_{rc}=\mathbf{P}_{rc}=\mathbf{LCB}_{rc}. \tag{21}$$

The potential energy during a pure rc-state consists of the strain energy $\mathbf{U}_{rc}$ and the load potential $\mathbf{W}_{rc}$:

$$\pi_{rc} = \mathbf{U}_{rc} + \mathbf{W}_{rc} = \frac{1}{2}\, \mathbf{q}_{rc}^T\mathbf{K}_{rc}\mathbf{q}_{rc} - \mathbf{q}_{rc}^T\mathbf{T}_{rc}\,. \tag{22}$$

Equilibrium requires that π_{rc} be stationary:

$$\delta\pi_{rc} = \delta\,\mathbf{q}_{rc}^T\mathbf{K}_{rc}\mathbf{q}_{rc} - \delta\,\mathbf{q}_{rc}^T\mathbf{T}_{rc} = 0\,. \tag{23}$$

By using Eq. (3), Eq. (10a), Eq. (18), Eq. (19) and $\mathbf{H}_{rc}\mathbf{G}_{rc}=\mathbf{I}$, it is evident that

$$\begin{aligned}\mathbf{K}_{rc}\mathbf{G}_{rc} &= V\mathbf{H}_{rc}^T\mathbf{B}_{rc}^T\mathbf{C}\mathbf{B}_{rc}\mathbf{H}_{rc}\mathbf{G}_{rc} \\ &= \mathbf{P}_{rc} = \mathbf{LCB}_{rc}\,.\end{aligned} \tag{24}$$

Thus,

$$\mathbf{L} = V\mathbf{H}_{rc}^T\mathbf{B}_{rc}^T\,. \tag{25}$$

This is equivalent to the following simple expression for the element stiffness matrix

$$\begin{aligned}\mathbf{K} &= \mathbf{K}_{rc} + \mathbf{K}_h \\ &= V\,(\mathbf{B}_{rc}\mathbf{H}_{rc})^T\, C\,(\mathbf{B}_{rc}\mathbf{H}_{rc}) + \mathbf{H}_h^T \int_V \mathbf{B}_h^T\,\mathbf{C}\mathbf{B}_h dV\mathbf{H}_h \\ &= \frac{1}{V}\mathbf{LCL}^T + \mathbf{H}_h^T \int_V \mathbf{B}_h^T\,\mathbf{C}\mathbf{B}_h dV\mathbf{H}_h \\ &= \frac{1}{V}\mathbf{LCL}^T + \mathbf{H}_h^T\mathbf{K}_{ah}\mathbf{H}_h\,.\end{aligned} \tag{26}$$

It is clear that **K** in Eq. (26) has no negative eigenvalues because it consists of quadratic forms on **C** and $\mathbf{K}_{ah}$ that are both positive definite.

The fundamental matrix **L** for a general quadrilateral plate bending element with sixteen degrees of freedom is given here by

$$\mathbf{L}=[\mathbf{L}_1\ \mathbf{L}_2\ \mathbf{L}_3\ \mathbf{L}_4], \tag{27}$$

where, typically,

Table 1. First Nine Frequencies of Vibration of a Clamped Square Plate

Eigenvalue Number	Computed Values	Theoretical Values Stokey (1961)
1	7.4620	7.4566
2	15 2148	15.2095
3	15.2148	15.2095
4	22.4668	22.4320
5	27.2300	27.2739
6	34.2206	34.2167
7	34.2206	34.2167
8	43.3433	-----
9	43.3433	-----

$$\mathbf{L}_j^T =$$

$$\frac{1}{2}\begin{bmatrix} 0 & y_i - y_k & 0 & (y_k - y_i)^2/6 \\ 0 & 0 & x_i - x_k & (x_k - x_i)^2/6 \\ 4(-1)^{(j-1)} & 0 & 0 & 0 \end{bmatrix}, \tag{28}$$

where the indices i and k are the nodes before and after node j, respectively, when going counterclockwise around the element.

IV. Element Mass Matrix

The natural frequencies of vibration of a plate system are governed by the following equation:

$$(\mathbf{K}_s - \omega^2 \mathbf{M}_s)\mathbf{q}_s = 0, \tag{29}$$

where ω are the natural frequencies of vibration in radians per second, and $\mathbf{K}_s$, $\mathbf{M}_s$ and $\mathbf{q}_s$ are the stiffness matrix, mass matrix and displacement vector of the whole system, respectively. The element stiffness matrix is given in Eq. (26). The element mass matrix is defined as

$$\mathbf{M} = \rho \int_V [\mathbf{N}_{rc}\mathbf{N}_h]^T [\mathbf{N}_{rc}\mathbf{N}_h] \, dV, \tag{30}$$

where ρ is the mass per unit volume of the element. $\mathbf{N}_{rc}$ and $\mathbf{N}_h$ are defined in Eq. (14) and Eq. (16), separatively. The subspace iteration method (Bathe, 1982) is employed to obtain the solution of the eigenvalue problems in Eq. (29).

V. Numerical Assessments

A computer program has been developed for numerical computation of various types of examples. All the computations have been per formed on an IBM personal computer system in double precision of real rounded arithmetic accuracy.

Except where specifically mentioned, the following values are adopted for analysis:

t(thickness of plate) = 0.05
E(Young's modulus of elasticity) = 3000.0
ν(Poisson's ratio) = 0.3
ρ(Mass density) = 1.0

Example 1: Rectangular Plates

Rectangular plates are widely used in the literature for assessing the performance of finite elements; therefore, for comparison purposes, they are also considered here. The frequencies of vibration are quoted in rad/sec, and a 2×2 Gaussian integrating rule is invariably employed.

The first case considered is that of a clamped square plate of side length 2.0 discretized into 16 elements. Table 1 shows the theoretical (Stokey, 1961) and computed values for the first nine frequencies of vibration. Close agreement can be observed. The eigenvectors for the first eight modes are shown in Fig. 2.

The next case considered is that of a square plate of side length 2.0 with two adjacent edges clamped. The theoretical (Stokey, 1961) and computed eigenvalues corresponding to the first nine frequencies are given in Table 2, and good agreement is also observed. The mode shapes for the first eight modes of vibration are shown in Fig. 3.

Some results for the free vibration of a simply supported square plate with regular and irregular meshes (Fig. 4) of side length 2.0 are shown in Table 3.

The computed eigenvalues of the first ten frequen-

Table 2. First Nine Frequencies of Vibration of a Clamped Square Plate with Two Adjacent Edgeds Clamped

Eigenvalue Number	Computed Values	Theoretical Values Stokey (1961)
1	1.4249	1.4416
2	4 8413	4.9890
3	5.4315	5.5526
4	9.6435	9.9553
5	12.5562	13.0817
6	13.2026	-----
7	17.0826	-----
8	17.7771	-----
9	23.9088	-----

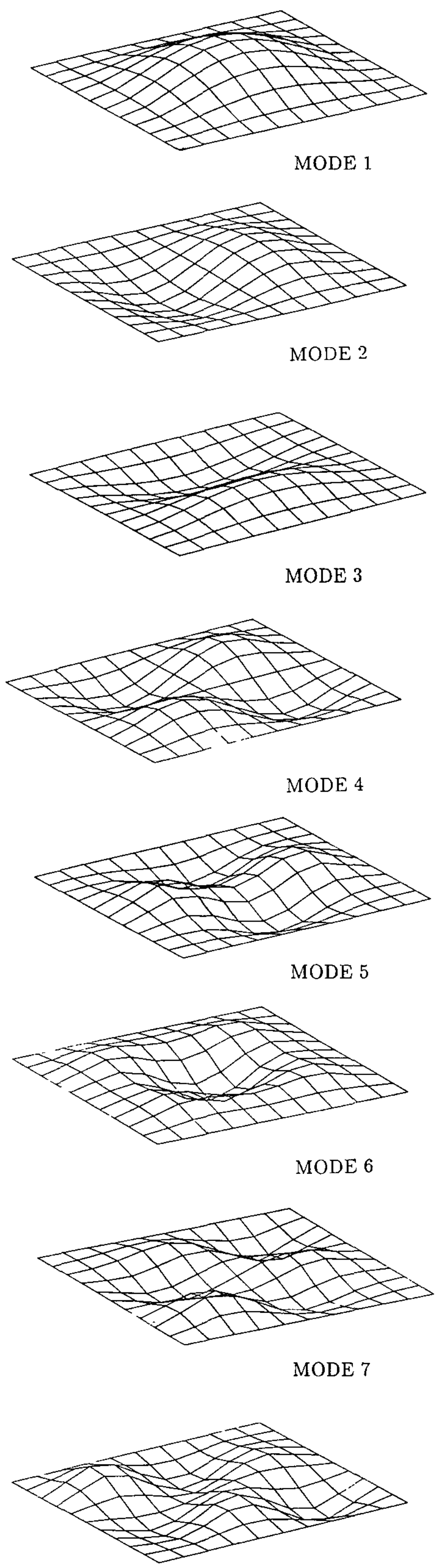

Fig. 2. First eight modes of vibration of a clamped square plate.

cies of vibration corresponding to the different patterns are listed, and it is seen that mesh 2 provides quite accurate results with respect to the regular mesh while results obtained from mesh 3 and mesh 4 are also consistent.

In Table 4, the convergence characteristics for the frequencies of a simply supported rec tangular plate of aspect ratio 2:1 are shown. The longer side length 2.0 is used, and results for the first eight modes of vibration are given. It is seen that aspect ratio has very little influence on the accuracy of the present method.

Example 2: Sector Plates

Free vibration analysis of annular sector plates is of practical and academic significance and has been studied by many authors. In Table 5, the first eight frequencies for a simply supported 90° sector plate with an inner to outer radius ratio of 0.5 are reported. In this case, a coarse mesh of 5x5 elements and a fine mesh of 10x10 elements were used for the analysis. Excellent agreement of the results obtained from the two solution schemes can be observed. For purposes of comparison, the

Table 3. First Nine Frequencies of Vibration of the Simply Supported Square Plates in Fig. 3

Eigenvalue Number	Mesh 1	Mesh 2	Mesh 3	Mesh 4	Exact
1	4.0962	4.0962	4.0535	4.0535	4.0897
2	10.2032	10.2032	7.3543	7.3543	10.2150
3	10.2032	10.2032	9.8910	9.8910	10.2150
4	16.2000	16.2000	11.3463	11.3463	16.3441
5	19.0843	19.0843	14.7262	14.7262	20.4301
6	19.0843	19.0843	15.5663	15.5663	20.4301
7	25.7420	25 7420	16.6860	16.6860	26.5591
8	25.7420	25.7420	18.3492	18.3492	26.5591
9	33.9495	33.9495	18.9976	18.9976	34.7311
10	33.9495	33.9495	26.5980	26 5980	34.7311

Table 4. First Eight Frequencies of Vibration of a Simply Support Rectangular Plate of Aspect Ratio 2:1

Eigenvalue Number	Element Mesh				Exact
	2x2	4x4	8x8	10x10	
1	9.2734	10.2236	10.2191	10.2185	10.2150
2	18.9487	16.2978	16.3565	16.3489	16.3441
3	19.3904	24.6112	26.5924	26.5680	26.5591
4	19.8652	34.1930	34.6878	34.6901	34.7311
5	20.2048	38.6344	40.7925	40.8121	40.8601
6	24.5775	43.2281	40.8136	40.8419	40.8601
7	25.2836	67.0834	50.9983	51.0257	51.0752
8	-----	67.1328	58.2506	58.9967	59.2473

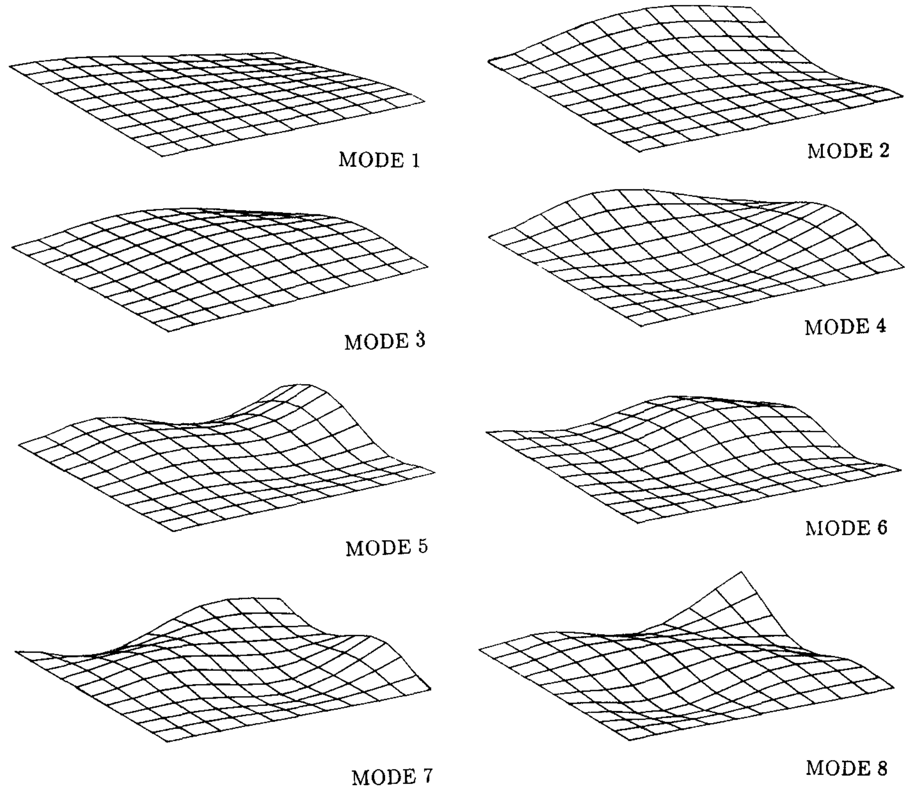

Fig. 3. First eight modes of vibration of a square plate with two adjacent edges clamped.

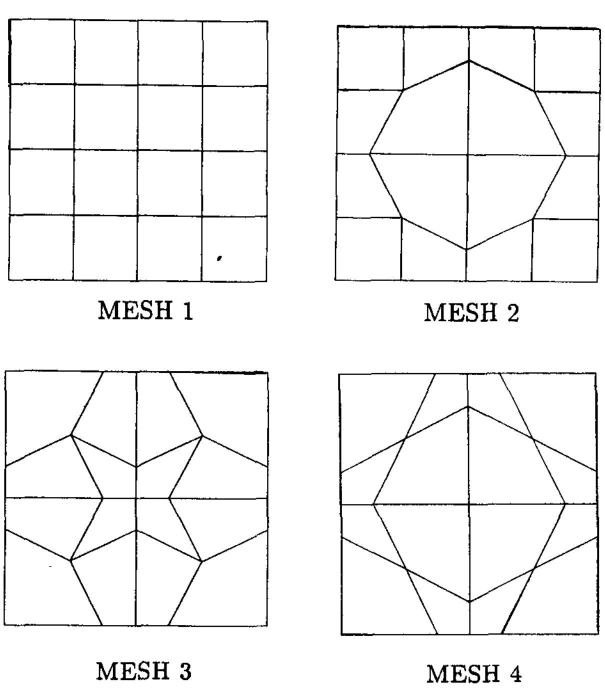

Fig. 4. Different meshes of a simply supported square plate.

results using a semi-analysis solution by Mukhopadhyay (1979) are also quoted, and again agreement between the results obtained by the two schemes is obvious. Plots of the mode shapes for this case are given in Fig. 5 for the first eight modes of free vibration.

Example 3: Cantilevered L-shaped Plate

The final example considered is that of the vibration of a cantilevered L-shaped plate shown in Fig. 6(a). The plate in Fig. 6(b) is discretized into 14 elements, and the material properties assumed for this problem are

t(thickness of plate) = 0.1
E(Young's modulus of elasticity) = 30×10^6
ν(Poisson's ratio) = 0.3
ρ(Mass density) = 0.00733

The lowest five natural frequencies are listed in Table 6, where they are compared with results from another quadrilateral plate model in Potts and Oler (1989). In this case, a coarse mesh of 14 elements and a fine mesh of 56 elements were used for the analysis. Excellent agreement of the results obtained from the two solution schemes can be observed. Plots of these mode shapes are shown in Fig. 7.

Table 5. First Eight Frequencies of Vibration of a 90° Sector Plate with all Edges Simply Supported with an Inner to Outer Radius Ratio of 0.5

Eigenvalue Number	Element 5×5	Mesh 10×10	Mukhopadhyay (1979)
1	9.8791	9.7628	9.7584
2	14.6683	14.3571	14.1306
3	22.0536	21.9794	21.3674
4	32.0732	31.5402	31.2046
5	34.9879	34.6039	34.3069
6	40.1197	39.7405	-----
7	43.8562	42.1744	-----
8	57.1977	56.6343	-----

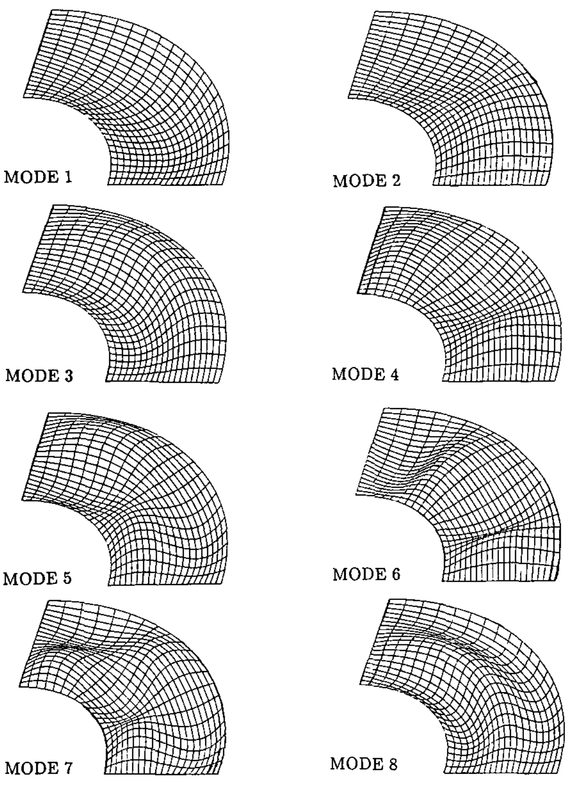

Fig. 5. First eight modes of vibration of a simply supported 90° sector plate.

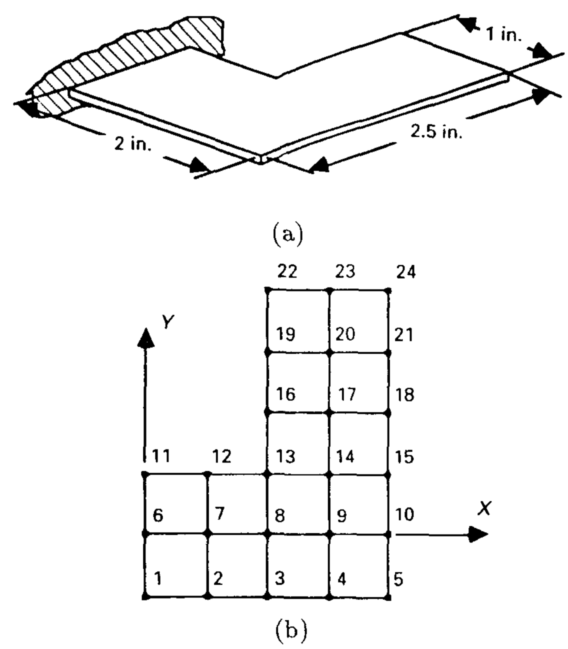

Fig. 6. (a) Cantilevered L-shaped plate. (b) Cantilevered L-shaped plate of 14 elements mesh.

Table 6. First Five Frequencies (Hertz) of Vibration of a Cantilevered L-Shaped Plate

Eigenvalue Number	14 elements	56 elements	Potts and Oler (1989)
1	435	438	435
2	929	977	970
3	2690	3137	3127
4	3571	3982	3990
5	5474	6229	6225

VI. Conclusions

The derivation of a general quadrilateral element with sixteen degrees of freedom for the free vibration analysis has been shown. The approach presented here for calculation of finite element solutions, based on a free formulation, opens up a number of avenues for further investigation. In this study, the shape functions were employed for the transverse displacement only. The relation between the corner rotations and the transverse displacement has been easily obtained by partial differential. The fundamental matrix **L** for the general quadrilateral element has been derived. The expressions obtained in the present formulation can be used comfortably in microcomputers.

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References

Bathe, K. J. (1982) *Finite Element Procedures in Engineering Analysis*. Prentice Hall, Englewood Cliffs, NJ, U.S.A.

Batoz, J. L., K. J. Bathe, and L. W. Ho (1980) A study of three-node triangular plate bending elements. *International Journal*

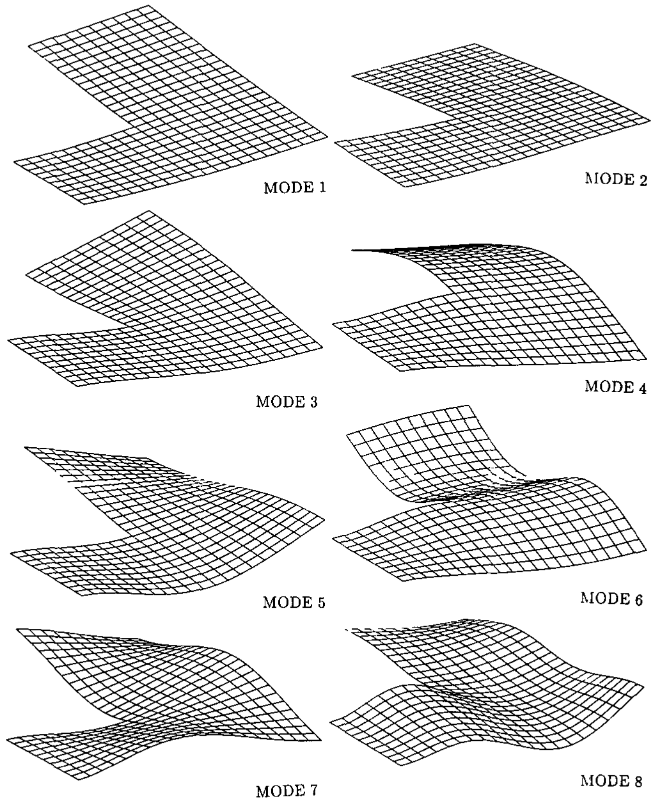

Fig. 7. First eight modes of vibration of a cantilevered L-shaped plate.

for Numerical Methods in Engineering, **15**, 1771-1812.

Batoz, J. L. and M. B. Tahar (1982) Evaluation of a new quadrilateral thin plate bending element. *International Journal for Numerical Methods in Engineering*, **18**, 1655-1677.

Bazeley, G. P., Y. K. Cheung, B. M. Irons, and O. C. Zienkiewicz (1965) Triangular elements in plate bending-conforming and non-conforming soultions. *Proc. 1st on Matrix Methods in Structural Mechanics*, pp. 547-576. Ohio, U.S.A.

Bergan, P. G. (1980) Finite elements based on energy orthogonal functions. *International Journal for Numerical Methods in Engineering*, **15**, 1541-1555.

Bergan, P. G. and L. Hanssen (1976) A new approach for deriving "good" finite elements. In: *The mathematics of finite elements and applications*, Vol. 2, J. R. Whiteman Ed. Academic Press, London, United Kindom.

Bergan, P. G. and M. K. Nygård (1981) Plate bending elements based on orthogonal functions. In: *New Concepts in Finite Element Analysis*, pp. 209-224. T. J. R. Hughes *et al.* Eds. ASME, **AMD-44**.

Bogner, F. K., R. L. Fox, and L. A. Schmit (1966) The generation of interelement compatible stiffness and mass matrices by the use of interpolation formulas. *Presented at the Wright-Patterson Air Force Base, Ohio Meeting on Matrix Methods in Structural Mechanics*, AFFDL-TR-66-80, 397-443.

Clough, R. W. and J. L. Tocher (1965) Finite element stiffness matrices for the analysis of plate bending. *Proc. 1st on Matrix Methods in Structural Mechanics*, pp. 547-576. Ohio, U.S.A.

Gallagher, R. H. (1969) Analysis of plate and shell structures. *Proc Symp. on Application of Finite Element Methods in Civil Engng.*, pp. 155-205. School of Engineering, Vanderbilt University, Nashville, Tennessee, U.S.A.

Hrabok, M. M. and T. M. Hrudey (1984) A review and catalogure of plate bending finite elements. *Computers and Structures*, **19**, 479-495.

Liu, I. W. and C. C. Lin (1993) A new conforming quadrilateral plate bending element. *International Journal for Numerical Methods in Engineering*, **36**, 2921-2937.

Mukhopadhyay, M. (1979) A semi-analysis solution for free vibration of annular sector plates. *Journal of Sound and Vibration*, **63**, 87-95.

Potts, J. F. and J. W. Oler (1989) *Finite Element Applications With Microcomputers*. Prentice Hall, Englewood Cliffs, NJ, U.S.A.

Stokey, W. F. (1961) Vibration of systems having distributed mass and elasticity. *Shock and Vibration Handbook*, Chapter 7. McGraw-Hill, New York, NY, U.S.A.

新的四節點任意四邊形板有限單元之自由振動分析

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摘　要

本文發展一個新的四節點任意四邊形彈性板有限單元之自由振動分析研究，其公式推導係根據Kirchhoff薄板理論與能量正交性理論，可以自動滿足收斂性要求。此彈性板有限單元之幾何外形是由雙線性多項式方程來模擬，其橫向位移函數是經由修正之雙立方曲線來表示，包含了一組基本剛體和等曲率變形模式加上一組高階變形模式。此四節點任意四邊形彈性板有限單元擁有十六個運動自由度，且與橫向位移函數同時建立在卡氏座標系上。同時本文針對典型之板結構實例探討不同之幾何外形效應下以及不規則網格劃分等，比較其差異性與精確性。

本研究不僅可以得到較經濟的四節點任意四邊形板有限單元，亦提供另一套有限單元之自由振動分析模式與方法。