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Inequality of Multiple Stochastic Integrals for Brownian Motion

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ABSTRACT

For a Brownian motion $B=(B_t)_{t\ge 0}$ with $B_0=0$, let $X=(X_t)_{t\ge 0}$ and $Y_n=(Y_n(t))_{t\ge 0}$ be two processes defined by

$$X_t=\int_0^t dB_{s_1}\int_0^{s_1} dB_{s_2}\cdots\int_0^{s_{n-1}} dB_{s_n} f(s_1,s_2,\ldots,s_n),$$

$$Y_n(t)=\left[\int_0^t ds_1\int_0^{s_1} ds_2\cdots\int_0^{s_{n-1}} ds_n f^2(s_1,s_2,\ldots,s_n)\right]^{1/2},$$

where $0<s_n<s_{n-1}<\ldots<s_2<s_1<t$, f is predictable in s_n and deterministic in the first $n-1$ variables. Also, assume $E(X_\infty^2)=E(Y_n^2(\infty))<\infty$.

The main result in this paper is:

For all $n\ge 1$, $0<p\le 1$, $0<b\le 2$, $b'>1$, there exists some constant C such that

$$E[\log(\sqrt{X_L^*}+1)^p]\le A_{b,n}+b^{-p}E[(\log(Y_n(L)+1))^p$$

$$E[\log(Y_n(L)+1)^p]\le A_{n,b'}+(b')^{-p}E[(\log(\sqrt{X_L^*}+1))^p$$

where $X_t^*=\sup_{s\le t}|X_s|$, L is an arbitrary random time and

$$A_{b,n}=nC+nC\frac{4ne}{2-b}2^{4/n}\exp\left(-\frac{2^{-4/n}(2-b)}{4ne}\right),$$

$$A_{n,b'}=ne^2+2^{4/n}\frac{n^2}{2(2b'-1)}\exp\left(\frac{2(1-2b')}{n}\right).$$

Key Words: Brownian motion, non-moderate function, multiple stochastic integrals

I. Introduction

Let $(\Omega, F(F_t)_{t\ge 0}, P)$ be a filtered probability space satisfying the usual condition, and $B=(B_t)_{t\ge 0}$ be a Brownian motion with respect to $(F_t)_{t\ge 0}$ where $B_0=0$.

The famous Burkholder-Davis-Gundy inequalities state that if F is a moderate function, there exist C_F, c_F such that

$$E(F(B_T^*))\le C_F E(F(\sqrt{T})), \qquad \textbf{(1)}$$

$$E(F(\sqrt{T}))\le c_F E(F(B_T^*)) \qquad \textbf{(2)}$$

for all (F_t) stopping times, T.

We recall that an increasing function F from $\Re_+$ to $\Re_+$ with $F(0)=0$ is called moderate if there exists $\alpha>1$ such that

$$\sup_{x>0}\frac{F(\alpha x)}{F(x)}<\infty.$$

It is interesting to ask how (1) and (2) might be modified to deal with non-moderate functions, such as the exponential function. The main idea is due to Jacka and Yor. In Jacka and Yor (1993), they found a way to solve it, and stated the following exponential type inequalities:

For $0<p<2$, there exist constants A, A', u, u' such that

$$E[\exp(B_L^*)^p] \leq A + uE[\exp\theta(\sqrt{L})^{\frac{2p}{2-p}}]$$

$$E[\exp(\sqrt{L})^p] \leq A' + u'E[\exp\theta'(B_L^*)^{\frac{2p}{2-p}}]$$

for an arbitrary random time L, if and only if $\theta > 2^{\frac{p}{2p}}$, $\theta' > (\frac{8}{\pi^2})^{\frac{p}{2-p}}$, respectively.

In Jacka and Yor (1993), they also showed the L^p-*norm* inequality for the couple $(X_\infty^*, Y_n(\infty))$. Our work in this paper is inspired by these two results. We will give an estimation of exponential type inequalities of multiple stochastic integrals for Brownian motion (see Propositions 5, 6). Moreover, the inequalities can also be established for some moderate functions, such as logarithmic type inequalities for $(B_t^*, t^{1/2})$ and (X^*, Y_n).

Next, we present some notations that we use throughout this paper.

Let $\mathcal{L}$ be the collection of random times, i.e. $\mathcal{L}=\{L$: L is an F-measurable, nonnegative random variable.$\}$, and $\mathcal{T}=\{T$: T is (F_t) stopping time.$\}$. Let (X, Y) be a pair of increasing, optional processes. Define

$P_L(x, y)=P(X_L \geq x, Y_L \leq y)$ for any $L \in \mathcal{L}$,

$P^*(x, y)=\sup_{T\in\mathcal{T}} P_T(x, y)$,

$P^*(t)=\sup_{y\geq 0} P^*(ty, y)$.

Jacka and Yor (1993) showed that for every x, $y \geq 0$,

$P^*(x, y)=\sup_{L\in\mathcal{L}} P_L(x, y)$.

II. Inequality for $(B_t^*, t^{\frac{1}{2}})$

The following lemma in Jacka and Yor (1993) is a step crucial to solving our problems in this paper.

Lemma 1: (Jacka and Yor's (1993) integral criterion).

Let F and G be two increasing, nonnegative, right continuous functions from R_+ to R_+. Define for each $m>0$

$$I_m = \int_0^\infty dx\, P^*\left(\frac{F^{-1}(x)}{G^{-1}(\frac{x}{m})}\right).$$

Let A_m be the best constant A appearing in

$E(F(X_L)) \leq A + mE(G(Y_L))$ for any $L \in \mathcal{L}$; **(3)**

then $A_m \leq I_m$, so that the inequality (3) holds for some A if $I_m < \infty$.

Although the purpose of the above lemma is to treat with non-moderate functions, in the next two propositions, we can still give two-sided inequalities for logarithm functions.

Proposition 1. For $0<p\leq 1$, $0<b<2$, there exists A_b such that

$$E[\log(\sqrt{B_L^*}+1)^p] \leq A_b + b^{-p}E[(\log(\sqrt{L}+1))^p]$$

for all $L \in \mathcal{L}$, $A_b = 2 + \frac{32}{2-b}\exp(-\frac{2-b}{16})$.

Proof: Define F, G: $\Re_+ \to \Re_+$ by

$$F(x) = (\log(x^{\frac{1}{2}}+1))^p,$$

$$G(x) = (\frac{1}{b}\log(x+1))^p;$$

then

$$F^{-1}(x) = (\exp(x^{1/p})-1)^2,$$

$$G^{-1}(x) = \exp(bx^{1/p})-1.$$

By the scaling property of Brownian motion, $P^*(x)$ for the couple $(B_t^*, t^{1/2})$ is given by $P^*(x)=P^*(B_1^* \geq x)$ $\leq 2P(|N| \geq x) \leq 2\exp(-\frac{1}{2}x^2)$, where N is a standard normal random variable.

Using an integral criterion, it is enough to calculate

$$I_1 = \int_0^\infty dx\, P^*\left(\frac{F^{-1}(x)}{G^{-1}(x)}\right).$$

For $0<b<2$, $0<p\leq 1$, $1\leq x<\infty$, we have

$$I_1 = \int_0^\infty dx\, P^*\left[\frac{(\exp(x^{1/p})-1)^2}{\exp(bx^{1/p})-1}\right]$$

$$\leq 2\int_0^1 1dx + 2\int_1^\infty dx \exp\left[-\frac{1}{2}\left(\frac{2^{-4}\exp 4x^{1/p}}{\exp 2bx^{1/p}}\right)\right]$$

$$\leq 2 + \frac{32}{2-b}\exp(-\frac{2-b}{16}).$$

The result follows.

Proposition 2. Suppose $0<p\leq 1$, $0<b'<\frac{1}{2}$; there exist $A_{b'}$ such that

$$E[\log(\sqrt{L}+1)^p] \leq A_{b'} + (b')^{-p}E[(\log(\sqrt{B_L^*}+1))^p]$$

where $L\in\mathcal{L}$ and $A_b = \frac{4}{\pi} + \frac{64}{\pi^3(1-2b')}\exp(-\frac{\pi^2(1-2b')}{16})$.

Proof: Define F, G: $\Re_+ \to \Re_+$ by

$$F(x) = (\log(x+1))^p,$$

$$G(x) = (\frac{1}{b'}\log(x^{\frac{1}{2}}+1))^p;$$

then

$$G^{-1}(x) = \exp(x^{\frac{1}{p}}) - 1,$$

$$F^{-1}(x) = (\exp(b'x^{\frac{1}{p}}) - 1)^2.$$

$P^*(x)$ for the pair $(t^{\frac{1}{2}}, B_t^*)$ is given by

$P^*(x) = P(T_1 \geq x^2)$, where $T_1 = inf\{t\geq 0: |B_t| = 1\}$.

It follows that

$$P^*(x) = \sum_{n=0}^{\infty}(-1)^n \frac{4}{(2n+1)\pi}\exp(-\frac{((2n+1)\pi)^2}{8}x^2)$$

$$\leq \frac{4}{\pi}\exp(-\frac{\pi^2}{8}x^2).$$

Since $0<p\leq 1$, $0<b'<\frac{1}{2}$, simple calculation gives

$$I_1 = \int_0^\infty dx\, P^*[\frac{(\exp(x^{1/p})-1)}{\exp((b'x^{1/p})-1)^2}]$$

$$\leq \frac{4}{\pi}\int_0^\infty dx \exp[-\frac{\pi^2}{8}(\frac{(\exp x^{1/p})-1}{\exp((b'x^{1/p})-1)^2})^2]$$

$$\leq \frac{1}{\pi}\int_0^1 1dx + \frac{4}{\pi}\int_0^\infty dx\exp[-\frac{\pi^2}{8}(\frac{\exp 2x^{1/p}}{\exp 4b'x^{1/p}})]$$

$$= \frac{4}{\pi} + \frac{64}{\pi^3(1-2b')}\exp(-\frac{\pi^2(1-2b')}{16}).$$

This completes the proof.

In the following, we are going to extend the inequalities for the couple $(B_t^*, t^{\frac{1}{2}})$ to multiple stochastic integrals for Brownian motions. Then, it can be seen that the main result of Jacka and Yor (1993) is a special case of Propositions 5 and 6.

III. Inequalities for $(X_\infty^*, Y_n(\infty))$

We consider the n-multiple stochastic integrals of the form

$$X_t = \int_0^t dB_{s_1}\int_0^{s_1} dB_{s_2}\cdots\int_0^{s_{n-1}} dB_{s_n} f(s_1, s_2, \ldots, s_n),$$

$$Y_n(t) = [\int_0^t ds_1\int_0^{s_1} ds_2\cdots\int_0^{s_{n-1}} ds_n f^2(s_1, s_2, \ldots, s_n)]^{1/2},$$

where $f(\ldots, *)$ is measurable in all its arguments, deterministic in the first $(n-1)$ time variables and is predictable in the n-th variable. Furthermore, we assume that $E(X_\infty^2) = E(Y_n^2(\infty)) < \infty$.

For $X=(X_t)_{t\geq 0}$, $Y_n=(Y_n(t))_{t\geq 0}$, we define these notations:

$$X_\infty = \int_0^\infty dB_{s_1} f_{n-1}(s_1)$$

$$= \int_0^\infty dB_{s_1}\int_0^{s_1} dB_{s_2} f_{n-2}(s_1, s_2)$$

$$= \int_0^\infty dB_{s_1}\int_0^{s_1} dB_{s_2}\cdots\int_0^{s_{n-2}} dB_{s_{n-1}} f_1(s_1, s_2, \ldots, s_{n-1})$$

$$\vdots$$

$$= \int_0^\infty dB_{s_1}\int_0^{s_1} dB_{s_2}\cdots\int_0^{s_{n-1}} dB_{s_n} f(s_1, s_2, \ldots, s_n),$$

$$Y_0(\infty) = |X_\infty|,$$

$$Y_1(\infty) = (\int_0^\infty ds_1 f_{n-1}^2(s_1))^{1/2},$$

$$\vdots$$

$$Y_n(\infty) = \int_0^\infty ds_1\int_0^{s_1} ds_2\cdots\int_0^{s_{n-1}} ds_n f^2(s_1, s_2, \ldots, s_n).$$

With abbreviation, we denote $Y_k(\infty)$ by Y_k for $0\leq k\leq n$; otherwise, if (X, Y) is a pair of increasing, optional processes, we also denote $P^*(t)$ by $P_{X,Y}^*(t)$. In fact, the integral criterion is based on the estimation of the upper bound for P^*. Jacka and Yor (1993) offered a method for finding the upper bound for P^*.

Lemma 2 (Jacka and Yor).

Suppose A and C are two increasing right continuous predictable processes with $A_0=C_0=0$ which satisfy

$$E(A_T^p) \leq a_p E(C_T^p)$$

for all $p>0$ and all stopping times T, where

$$a_p \le K(c_0 + d_0 p)^{c_1 + d_1 p};$$

then

$$P^*(t) \le K\left(\frac{d_0}{d_1}\right)^{c_1} u^{c_1} \exp(b - c_1) \exp\left(-\frac{u}{e}\right),$$

where $u \equiv u(t) = \frac{d_0}{d_1} t^{\frac{1}{d_1}}$, and $b = \frac{c_0 d_1}{d_0}$.

Lemma 3. For $C, c, , \beta > 0$, if $Y_0, Y_1, \ldots, Y_n$ are defined as above and satisfy

(1) $P^*_{Y_k^*, Y_{k+1}}(t) \le C \exp(-ct^\beta)$, then

$P^*_{Y_0^*, Y_n}(t) \le nC \exp(-ct^{\frac{\beta}{n}})$.

(2) $P^*_{Y_{k+1}, Y_k^*}(t) \le t^\beta \exp(-ct^\beta)$, then

$P^*_{Y_n, Y_0^*}(t) \le nt^{\frac{\beta}{n}} \exp(-ct^{\frac{\beta}{n}})$.

Proof: (1) By induction, assume that the result holds for $n-1$; then for any $y>0$, and $x, t \ge 0$, we have

$$P(Y_0^* \ge xt, Y_n < x)$$

$$\le P(Y_0^* \ge xt, Y_{n-1} < y) + P(Y_{n-1} \ge y, Y_n < x)$$

$$\le (n-1) C \exp\left(-c\left(\frac{xt}{y}\right)^{\frac{\beta}{n-1}}\right) + C \exp\left(-c\left(\frac{x}{y}\right)^\beta\right).$$

Setting $y = xt^{1/n}$, we obtain

$$P(Y_0^* \ge xt, Y_n < x) \le nC \exp(-ct^{\frac{n}{\beta}}).$$

The result follows:

(2) $P(Y_n \ge xt, Y_0^* < x)$

$$\le P(Y_n \ge xt, Y_{n-1} < y) + P(Y_{n-1} \ge y, Y_0^* < x)$$

$$\le \left(\frac{xt}{y}\right)^\beta \exp\left(-c\left(\frac{xt}{y}\right)^\beta\right)$$

$$+ (n-1)\left(\frac{x}{y}\right)^{\frac{\beta}{n-1}} \exp\left(-c\left(\frac{x}{y}\right)^{\frac{\beta}{n-1}}\right).$$

Setting $y = xt^{\frac{n-1}{n}}$, the result follows again.

Now we could use Lemma 2, Lemma 3 and the following results that Jacka and Yor (1993) showed to get the upper bounds of P^*. Jacka and Yor (1993) stated that for all $0 \le m \le n-1$, $p > 0$,

$$E[(Y_m^*(T))^p] \le a_p' E[(Y_{m+1}(T))^p], \tag{4}$$

$$E[(Y_{m+1}(T))^p] \le a_p E[(Y_m^*(T))^p] \tag{5}$$

for all stopping times T, where $a_p' \le C(4(p+\frac{1}{2}))^{\frac{p}{2}}$ for some constant C, $a_p \le (e(p+\frac{1}{2}))^{1+\frac{p}{2}}$.

Applying Lemma 2 to the inequalities (4) and (5), we get

$$P^*_{Y_k^*, Y_{k+1}}(t) \le C \exp\left(-\frac{t^2}{8e}\right).$$

$$P^*_{Y_{k+1}, Y_k^*}(t) \le t^2 \exp\left(-\frac{t^2}{e^2}\right).$$

Then Lemma 3 yields

$$P^*_{Y_0^*, Y_n}(t) \le nC \exp\left(-\frac{t^{\frac{2}{n}}}{8e}\right). \tag{6}$$

$$P^*_{Y_n, Y_0^*}(t) \le t^2 \exp\left(-\frac{t^{\frac{2}{n}}}{e^2}\right). \tag{7}$$

The integral criterion of Lemma 1 relys essentially upon the upper bound of P^*. Hence, these two upper bounds (6), (7) in the above could lead to the following results.

Proposition 3. For all $n \ge 1$, $0 < p \le 1$, $0 < b \le 2$, there exists $A_{b,n}$ such that

$$E[(\log(\sqrt{X_L^*} + 1))^p]$$

$$\le A_{b,n} + b^{-p} E[(\log(Y_n(L) + 1))^p],$$

where $A_{b,n} = nC + nC \frac{4ne}{2-b} 2^{\frac{4}{n}} \exp\left(-\frac{2^{\frac{-4}{n}}(2-b)}{4ne}\right)$.

Proof: We define F, G, and continue the proof by using (6): $P^*_{Y_0^*, Y_n}(t) \le nC \exp\left(-\frac{t^{2/n}}{8e}\right)$ as in Proposition 1. This leads to the following result:

Proposition 4. For all $n \ge 1$, $0 < p \le 1$, $b' > 1$, there exists $A_{n,b'}$ such that

$$E[\log(Y_n(L) + 1)^p]$$

$$\le A_{n,b'} + (b')^{-p} E[(\log(\sqrt{X_L^*} + 1))^p],$$

where $A_{n,b'} = ne^2 + 2^{4/n} \frac{n^2}{2(2b'-1)} \exp\left(\frac{2(1-2b')}{n}\right)$.

Proof: Define F, G: $\mathfrak{R}_+ \to \mathfrak{R}_+$ by

$$F(x) = (\log(x+1))^p,$$

$$G(x)=(\frac{1}{b'}\log(\sqrt{x}+1))^{p};$$

then

$$F^{-1}(x)=\exp(x^{\frac{1}{p}})-1,$$

$$G^{-1}(x)=(\exp(b'x^{\frac{1}{p}})-1)^{2}.$$

By (7): $P^*_{Y_n,X^*}(x)\le n x^{\frac{2}{n}}\exp(-\frac{x^{\frac{2}{n}}}{e^2})$, we have

$$I_1=\int_0^\infty dxP^*(\frac{(\exp(x^{\frac{1}{p}})-1)}{\exp((b'x^{\frac{1}{p}})-1)^{2}})$$

$$\le n\int_0^\infty dx(\frac{(\exp(x^{\frac{1}{p}})-1)}{\exp((b'x^{\frac{1}{p}})-1)^{2}})^{\frac{2}{n}}$$

$$\bullet\exp(-\frac{1}{e^2}\frac{(\exp(x^{\frac{1}{p}})-1)^{\frac{2}{n}}}{(\exp(b'x^{\frac{1}{p}})-1)^{\frac{4}{n}}}).$$

Define $f(x)=x\exp(-\frac{x}{e^2})$ for $x\ge0$; the first derivative of f issues that $f(x)$ is dominated by e^2. For $b'>1$, $x\ge1$, $\frac{1}{p}\ge1$, the inequality becomes

$$I_1\le n\int_0^1 e^2\,dx+n\int_1^\infty \exp(\frac{(\exp(x^{\frac{1}{p}})-1)^{\frac{2}{n}}}{(\exp(b'x^{\frac{1}{p}})-1)^{\frac{4}{n}}})$$

$$\le ne^2+n\int_1^\infty dx2^{\frac{4}{n}}\exp(\frac{2(1-2b')x}{n})$$

$$=ne^2+2^{\frac{4}{n}}\frac{n^2}{2(2b'-1)}\exp(\frac{2(1-2b')}{n}).$$

The result follows.

Actually, Propositions 3 and 4 give us the two-sided logarithmic type inequalities for the pair (X^*_∞, $Y_n(\infty)$). Furthermore, they are the extensions of Propositions 1 and 2. As we choose n=1 and f=1 in Propositions 3, 4, and compare it to Propositions 1, 2, we could have the different constants between them. Naturally, this is due to different estimations for P^*.

Finally, we are going to give the generalization of Jacka-Yor's exponential type inequality for Brownian motion.

Proposition 5. For all $n\ge1$, $0<p<2$, there exists $A_{n,b}$ such that

$$E[\exp(X^*_L)^{\frac{p}{n}}]\le A_{n,b}+E[\exp(bY_n(L))^{\frac{2p}{n(2-p)}}],$$

where $A_{n,b}=nC(1+(\frac{1}{8e}b^{\frac{2-p}{p}}-1)^{-1})$, provided $b>(8e)^{\frac{p}{2-p}}$.

Proof: Define F, G: $\Re_+\to\Re_+$ by

$$F(x)=\exp(x^{\frac{p}{n}}),$$

$$G(x)=\exp(bx^{\frac{2p}{n(2-p)}});$$

then

$$F^{-1}(x)=(\log x)^{\frac{n}{p}},$$

$$G^{-1}(x)=b^{-\frac{n(2-p)}{2p}}(\log x)^{\frac{n(2-p)}{2p}}.$$

By the integral criterion, we have

$$I_1=\int_0^\infty dxP^*(b^{-\frac{n(2-p)}{2p}}(\log x)^{\frac{n}{2}}).$$

Using (6): $P^*_{Y^*_0,Y_n}\le nC\exp(-\frac{t^{\frac{2}{n}}}{8e})$, and some simple estimations and calculations can give

$$I_1\le nC(\int_0^1 dx+\int_1^\infty dx\exp(-\frac{1}{8e}b^{\frac{2-p}{p}}(\log x)))$$

$$\le nC(1+(\frac{1}{8e}b^{\frac{2-p}{p}}-1)^{-1})$$

provided $b^{\frac{2-p}{p}}>8e$. This completes the proof.

Proposition 6. For all $n\ge1$, $0<p<2$, there exist $A_{n,b'}$ such that

$$E[\exp(Y_n(L))^{\frac{p}{n}}]\le A_{n,b'}+E[\exp(b'X^*_L)^{\frac{2p}{n(2-p)}}],$$

where $A_{n,b'}=n(1+(e^{-2}b'^{\frac{2-p}{p}}-1)^{-2}b'^{\frac{2-p}{p}})$, provided $b'>\exp(\frac{2p}{2-p})$.

Proof: Define F, G as in Proposition 5, by (7): $P^*_{Y_n,Y^*_0}\le P^*_{Y_n,Y^*_0}\le nt^{\frac{2}{n}}\exp(-\frac{t^{\frac{2}{n}}}{e^2})$, and we have

$$I_1\le n(\int_0^1 dx+\int_1^\infty dxP^*(b'^{\frac{n(2-p)}{2p}}(\log x)^{\frac{n}{2}}))$$

$$\le n(1+\int_1^\infty dxb'^{\frac{(2-p)}{p}}x^{-e^{-2}b'^{\frac{2-p}{p}}}\log x)$$

$$=n\left(1+\left(e^{-2}b'^{\frac{2-p}{p}}-1\right)^{-2}b'^{\frac{p}{2-p}}\right)$$

if $0<p<2$, $b'^{\frac{2-p}{p}}>e^2$.
This finishes the proof.

In fact, Theorem 1 in Jacka and Yor (1993) is the particular case of Propositions 5 and 6 if we choose n=1, and f=1 in Propositions 5 and 6. As we choose n=1 and f=1 in Propositions 5 and 6, we compare it with Theorem 1 in Jacka and Yor (1993), and the differences between them are the constants A and $A_{1,b}$, A' and $A_{1,b'}$. Obviously, this is caused by the different estimations for P^*.

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布朗運動多重隨機積分不等式

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摘　要

藉由一個布朗運動$B=(B_t)_{t\geq 0}$其$B_0=0$，定義兩個隨機過程$X=(X_t)_{t\geq 0}$，$Y_n=(Y_n(t))_{t\geq 0}$

$$X_t=\int_0^t dB_{s_1}\int_0^{s_1} dB_{s_2}\cdots\int_0^{s_{n-1}} dB_{s_n} f(s_1,s_2,\ldots,s_n),$$

$$Y_n(t)=\left[\int_0^t ds_1\int_0^{s_1} ds_2\ldots\int_0^{s_{n-1}} ds_n f^2(s_1,s_2,\ldots,s_n)\right]^{1/2},$$

這裡$0<s_n<s_{n-1}<\ldots<s_2<s_1<t$，$f$在第$s_n$的時間是可預測的，在前$(n-1)$個時間是決定性的，並且滿足$E(X_\infty^2)=E(Y_n^2(\infty))<\infty$.

這篇論文的主要結果是：

對於任意的$n\geq 1, 0<p\leq 1$，$0<b\leq 2$，$b'>1$，存在$A_{b,n}$，$A_{n,b}$使得

$$E[\log(\sqrt{X_L^*}+1)^p]\leq A_{b,n}+b^{-p}E[(\log(Y_n(L)+1))^p],$$

$$E[\log(Y_n(L)+1)^p]\leq A_{n,b'}+(b')^{-p}E[(\log(\sqrt{X_L^*}+1))^p],$$

這裡的L是任意的隨機時間，C是某一常數，且

$$A_{b,n}=nC+nC\frac{4ne}{2-b}2^{4/n}\exp\left(-\frac{2^{-4/n}(2-b)}{4ne}\right),$$

$$A_{n,b'}=ne^2+2^{4/n}\frac{n^2}{2(2b'-1)}\exp\left(\frac{2(1-2b')}{n}\right).$$