

(Short Communication)

# On the Hole Effect of Image Rotation Algorithms

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## ABSTRACT

This paper proposes a mathematical model demonstrating that holes may be generated in images rotated through certain angles while for other rotation angles, no holes are generated. The hole effect is a classic problem in conventional image rotation, but the mechanism giving rise to the effect is not at all obvious. For many image-rotation algorithms, researchers have asserted that such holes may be simply treated as a side effect. A number of individuals have discussed hole-free rotation algorithms, but few have effectively explained why such holes are generated. A total of eight definitions, six theorems, and related experiments are used here to explicate clearly the mechanism producing such image rotation holes. For those individuals who wish to develop new hole-free rotation algorithms, the model presented herein can play an important role as a future reference.

**Key Words:** associative function, hole effect, ideal rotation, truncation, round-off, post-integer-conversion, rotation

## I. Introduction

Rotation is a fundamental geometric transformation used in computer graphics and image processing. The rotation of an image requires the calculation of a new position for each point of the image after the transformation. The method used in most textbooks (Foley *et al.*, 1990; Hearn and Baker, 1994; Hill, 1990; Plastock and Kalley, 1986) and by many researchers (Cheng *et al.*, 1990; Chien and Baek, 1998) for rotating images is briefly reviewed as follows.

An image point  $p = (x, y)$  on the x-y plane is rotated about the origin of the x-y plane through use of the following transformation:

$$x' = x \cos\theta - y \sin\theta, \quad (1a)$$

$$y' = x \sin\theta + y \cos\theta, \quad (1b)$$

where  $p' = (x', y')$  is the associated point after rotation, and  $\theta$  is the angle of rotation. Figure 1 shows the geometric relationship between  $p$ ,  $p'$  and  $\theta$ .

In raster-scan displays, each rotated point is first calculated by using Eq. (1), and a truncation or a round-off

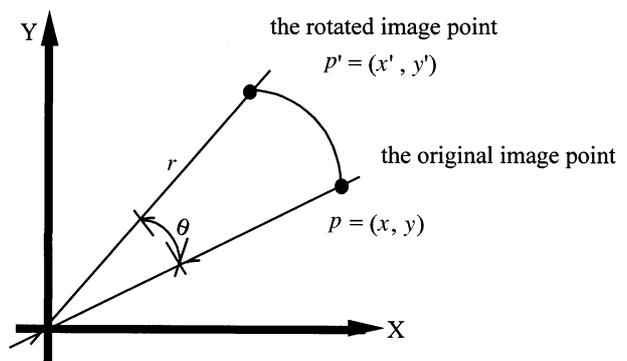


Fig. 1. An image point rotated through an angle  $\theta$  about the origin.

function is then used to get the integer values of its calculated floating point coordinates. We will denote the truncation or the round-off as the post-integer-conversion. Using Eq. (1), the rotation of a rectangular image array  $A_{mn}$  of image points may be described as follows:

```

for x = 0 to m-1
begin
    for y = 0 to n-1
    begin
    
```

```

NewX = x cosθ - y sinθ
NewY = x sinθ + y cosθ
NewColor = col_table(x,y)
put_pixel(NewX, NewY, NewColor)
end
end
    
```

where  $m$  is the width of the original image,  $n$  is the height of the original image, and  $col\_table$  stores the color attribute of the original image.

However, use of the above image rotating algorithm, for some values of  $\theta$ , can result in the creation of holes, which was referred to as the “measles” problem by Cheng *et al.* (1990), while for some rotation angles, no holes are generated.

In this paper, the authors propose a rigorous mathematical model to define and demonstrate various hole effects. The remainder of the paper is organized as follows. Section II presents the mathematical model. Section III presents discussion of the various hole effects based on our proposed model, and also presents experimental results which support our model. Finally, we provide a set of short conclusions in Section IV.

## II. The Model of the Hole Effect

Definitions of some related terminology are as follows:

**Definition 1. Plane P.**  $P$  is a plane consisting of a set of points  $p \equiv (x,y)$ , where  $x, y \in R \equiv$  the set of real numbers.

**Definition 2. The round-off function ROUND.** The round-off function ROUND is defined as  $ROUND(f) = n$ , where  $n \leq f + 1/2 < n + 1$ ,  $f \in R$  and  $n \in Z \equiv$  the set of integers.

**Definition 3. The truncation function TRUNCATE.** The truncation function TRUNCATE is defined as  $TRUNCATE(f) = n$ , where  $n \leq f < n + 1$ ,  $f \in R$  and  $n \in Z$ .

**Definition 4. The associative function ASSOC.**

$$ASSOC(p_i, p_j) = \begin{cases} 0, & \text{if } |p_i - p_j| \\ & = \sqrt{|x_i - x_j|^2 + |y_i - y_j|^2} \leq \sqrt{2}; \\ 1, & \text{otherwise.} \end{cases} \quad (2)$$

$\forall i, j \in I \equiv$  the set of nonnegative integers, and  $p_i, p_j \in P$ .

**Definition 5. A 4NAP.** The four distinct points  $p_0, p_1, p_2$ , and  $p_3$  are said to form a 4NAP (Four Near Associative Points) if  $ASSOC(p_i, p_j) = 0, \forall i, j, 0 \leq i \leq 3, 0 \leq j \leq 3$ , and  $i \neq j$ .

**Definition 6. A 4FAP.** The four distinct points  $p_0, p_1, p_2$ , and  $p_3$  are said to form a 4FAP (Four Far Associative Points) if  $ASSOC(p_0, p_1) = ASSOC(p_1, p_2) = ASSOC(p_2, p_3) = ASSOC(p_3, p_0) = 0$  and  $ASSOC(p_0, p_2) = ASSOC(p_1, p_3) = 1$ .

**Definition 7. Hole.** If the five distinct points  $p_h, p_0, p_1, p_2$ , and  $p_3$  are such that  $p_0, p_1, p_2$  and  $p_3$  form a 4FAP, and if  $p_h$  is surrounded by this 4FAP, then  $p_h$  is a hole.

An image rotation is an Ideal Rotation (I.R.) if there is no hole in the rotated image. Using the above terminology, we will discuss the hole effect in the following.

**Definition 8. Ideal Rotation.** Let  $p_0, p_1, p_2$  and  $p_3$  form a 4NAP, as shown in Fig. 2, where  $p_0 = (x, y)$ ,  $p_1 = (x + 1, y)$ ,  $p_2 = (x + 1, y + 1)$ ,  $p_3 = (x, y + 1)$ , and  $x, y \in P$ . Using Eq. (1), the four corresponding rotated points are calculated as follows:

$$\begin{aligned} p'_0 &= (x'_0, y'_0) \\ &= ((x \cos \theta - y \sin \theta), (x \sin \theta + y \cos \theta)), \end{aligned} \quad (3)$$

$$\begin{aligned} p'_1 &= (x'_1, y'_1) \\ &= (((x + 1) \cos \theta - y \sin \theta), ((x + 1) \sin \theta + y \cos \theta)), \end{aligned} \quad (4)$$

$$\begin{aligned} p'_2 &= (x'_2, y'_2) \\ &= (((x + 1) \cos \theta - (y + 1) \sin \theta), \\ & \quad ((x + 1) \sin \theta + (y + 1) \cos \theta)), \end{aligned} \quad (5)$$

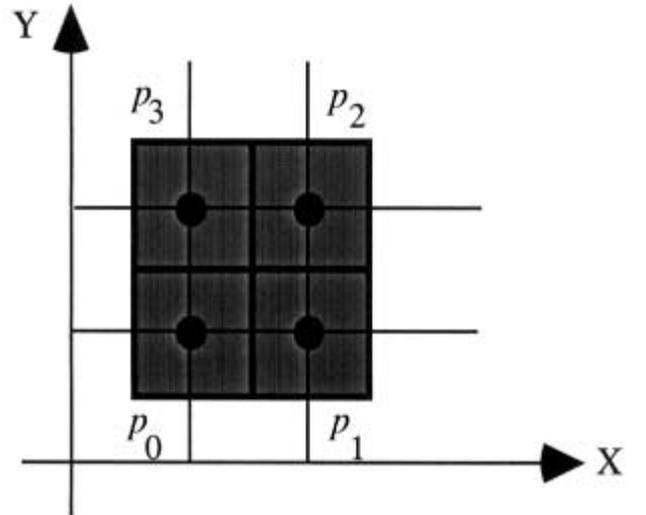


Fig. 2. The four distinct points  $p_0, p_1, p_2$ , and  $p_3$  form a 4NAP.

$$\begin{aligned}
p'_3 &= (x'_3, y'_3) \\
&= ((x \cos \theta - (y+1) \sin \theta), (x \sin \theta + (y+1) \cos \theta)).
\end{aligned} \tag{6}$$

Since the calculated values of the  $x$ 's and the  $y$ 's are floating-point numbers, it is necessary to use either truncation or round-off to convert these into integers (i.e., post-integer-conversion) in order to organize the related pixels on the screen.

The values of the associative function are essential to the proof of the hole effect. Equation (1) is used to calculate some of these as follows:

$$\begin{aligned}
|x'_1 - x'_0| &= |(x+1) \cos \theta - y \sin \theta - x \cos \theta + y \sin \theta| \\
&= |\cos \theta| \leq 1,
\end{aligned} \tag{7a}$$

$$\begin{aligned}
|y'_1 - y'_0| &= |(x+1) \sin \theta + y \cos \theta - x \sin \theta - y \cos \theta| \\
&= |\sin \theta| \leq 1,
\end{aligned} \tag{7b}$$

$$\Rightarrow \sqrt{|x'_1 - x'_0|^2 + |y'_1 - y'_0|^2} \leq \sqrt{2},$$

$$\Rightarrow \text{ASSOC}(p'_0, p'_1) = 0,$$

$$\begin{aligned}
|x'_2 - x'_1| &= |(x+1) \cos \theta - (y+1) \sin \theta - (x+1) \cos \theta \\
&\quad + y \sin \theta| = |-\sin \theta| \leq 1,
\end{aligned} \tag{8a}$$

$$\begin{aligned}
|y'_2 - y'_1| &= |(x+1) \sin \theta + (y+1) \cos \theta - (x+1) \sin \theta \\
&\quad - y \cos \theta| = |\cos \theta| \leq 1,
\end{aligned} \tag{8b}$$

$$\Rightarrow \text{ASSOC}(p'_1, p'_2) = 0,$$

$$\begin{aligned}
|x'_3 - x'_2| &= |x \cos \theta - (y+1) \sin \theta - (x+1) \cos \theta \\
&\quad + (y+1) \sin \theta| = |-\cos \theta| \leq 1,
\end{aligned} \tag{9a}$$

$$\begin{aligned}
|y'_3 - y'_2| &= |x \sin \theta + (y+1) \cos \theta - (x+1) \sin \theta \\
&\quad - (y+1) \cos \theta| = |-\sin \theta| \leq 1,
\end{aligned} \tag{9b}$$

$$\Rightarrow \text{ASSOC}(p'_2, p'_3) = 0.$$

$$\begin{aligned}
|x'_0 - x'_3| &= |x \cos \theta - y \sin \theta - x \cos \theta + (y+1) \sin \theta| \\
&= |\sin \theta| \leq 1,
\end{aligned} \tag{10a}$$

$$\begin{aligned}
|y'_0 - y'_3| &= |x \sin \theta + y \cos \theta - x \sin \theta - (y+1) \cos \theta| \\
&= |-\cos \theta| \leq 1, \\
\Rightarrow \text{ASSOC}(p'_3, p'_0) &= 0.
\end{aligned} \tag{10b}$$

To complete the mathematical model, various hole effects are divided into the following types, with some post-integer-conversion explained:

- (1)  $R_{135t}$  stands for rotation of  $135^\circ$ , and post-integer-conversion is used to truncate the floating-point value.
- (2)  $R_{135r}$  stands for rotation of  $135^\circ$ , and post-integer-conversion is used to round off the floating-point value.
- (3)  $R_{\theta r}$  stands for rotation through a random angle ( $\theta$ ), and post-integer-conversion is used to round off, where  $\theta$  can be any angle except a multiple of  $90^\circ$ .
- (4)  $R_{\theta t}$  stands for rotation through a random angle ( $\theta$ ), and post-integer-conversion is used to truncate, where  $\theta$  can be any angle except a multiple of  $90^\circ$ .
- (5)  $R_{st}$  stands for rotation through a special angle (i.e., a multiple of  $90^\circ$ ), and post-integer-conversion is used to round-off.
- (6)  $R_{sr}$  stands for rotation through a special angle (i.e., a multiple of  $90^\circ$ ), and post-integer-conversion is used to truncate.

### III. Proof of the Various Hole-Effect Theorems

**Theorem 1.** *The rotation  $R_{135t}$  is not an I.R.*

*Proof.* By Definitions 6 and 7,  $R_{135t}$  is not an I.R. if there exists a hole surrounded by  $p'_0, p'_1, p'_2$  and  $p'_3$  as shown in Fig. 3, i.e., if

$$\text{ASSOC}(p'_0, p'_2) = 1, \text{ASSOC}(p'_1, p'_3) = 1,$$

and

$$\begin{aligned}
&\text{ASSOC}(p'_0, p'_1) = \text{ASSOC}(p'_1, p'_2) \\
&= \text{ASSOC}(p'_2, p'_3) = \text{ASSOC}(p'_3, p'_0) = 0
\end{aligned} \tag{11}$$

$$\forall i, j, 0 \leq i \leq 3, 0 \leq j \leq 3, \text{ and } i \neq j.$$

In Eq. (11), the last four equalities have already been derived in Eqs. (7)-(10) and need not be mentioned again. It, therefore, remains to be shown that:

$$\text{ASSOC}(p'_0, p'_2) = 1 \tag{12}$$

and

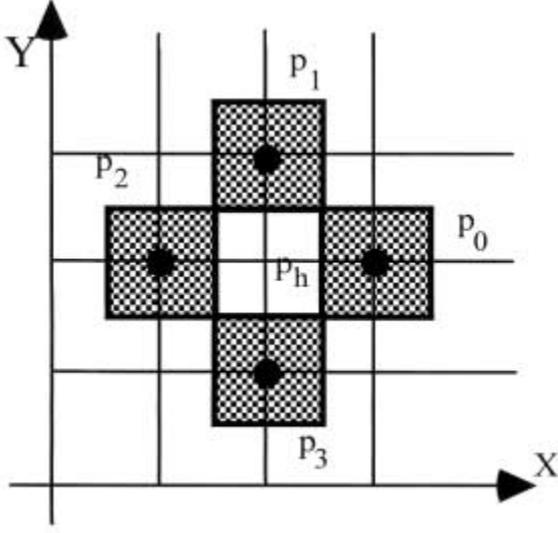


Fig. 3. The four distinct points  $p_0, p_1, p_2,$  and  $p_3$  form a 4FAP, and  $p_h$  is a hole.

$$\text{ASSOC}(p'_1, p'_3) = 1 \quad (13)$$

Condition (12) can be divided into Case 1:  $x'_2 > x'_0$  and Case 2:  $x'_2 < x'_0$ . Similarly, condition (13) can be split into Case 3:  $y'_3 < y'_1$  and Case 4:  $y'_3 > y'_1$ .

Note that for condition (12), the magnitude of  $|y'_0 - y'_2|$  is not essential. This can be seen by noting that

$$\begin{aligned} |x'_0 - x'_2| &= |x \cos \theta - y \sin \theta - (x+1) \cos \theta + (y+1) \sin \theta| \\ &= |\sin \theta - \cos \theta| \end{aligned} \quad (14a)$$

$$\begin{aligned} |y'_0 - y'_2| &= |x \sin \theta + y \cos \theta - (x+1) \sin \theta - (y+1) \cos \theta| \\ &= |\sin \theta + \cos \theta|. \end{aligned} \quad (14b)$$

Therefore, with  $\theta = 135^\circ$ , we have  $|\sin \theta + \cos \theta| = 0$ ,  $|\sin \theta - \cos \theta| > 1$ . Since  $|\sin \theta + \cos \theta| = 0$ , the term  $|y'_0 - y'_2|$  clearly has nothing to do with the associative function. For  $\text{ASSOC}(p'_0, p'_2) = 1$ , only the term  $|x'_0 - x'_2|$  needs be considered.

**Case 1**  $x'_2 > x'_0$ . From Eq. (14), it follows that

$$x'_2 = x'_0 + |\sin \theta + \cos \theta| \text{ with } \theta = 135^\circ. \quad (15)$$

When  $\theta = 135^\circ$ , values for  $\theta_{\min}$ ,  $\theta_{\max}$ , and  $\beta$  are such that  $\theta_{\min} \leq \theta \leq \theta_{\max}$ ,  $|\sin \theta - \cos \theta| = 1 + \beta$ , and  $0 \leq \beta < 0.5$ . For example, let  $\theta = 135^\circ$ ,  $\theta_{\min} = 125^\circ$ , and  $\theta_{\max} = 145^\circ$ ; then  $\beta = \sqrt{2} - 1$  (or 0.414...), so  $0 \leq \beta < 0.5$ , and condition (12) is obviously true. To help the reader understand the above theory, another example is used to provide greater insight into how the theory works.

Taking  $\theta = 135^\circ$ , the rotated coordinate  $x'_0$  of  $x_0$  is  $x \cos \theta - y \sin \theta$ , which is a floating-point number. Any floating-point number can be represented as the sum of its integer part and its decimal part; thus, the floating-point value  $x'_0$  is given by

$$x'_0 = n + \alpha, \quad (16)$$

where  $n (\in \mathbb{Z})$  is the integer part and  $\alpha (\in \mathbb{R}, 0 \leq \alpha < 1)$  is the decimal part, respectively. Applying Eq. (1), it follows that

$$x \cos \theta - y \sin \theta = n + \alpha \quad (17)$$

or

$$\alpha = x \cos \theta - y \sin \theta - n. \quad (18)$$

For given values of  $\theta = 135^\circ$  and  $\alpha$ , it is easily seen that there are many sets of values of  $x, y$  and  $n$  satisfying Eq. (18), assuming that one of the many sets making Eq. (18) is valid. Using this set, with  $\theta = 135^\circ$  and with  $\alpha$  set to 0.7, Eq. (18) becomes

$$\alpha = x \cos \theta - y \sin \theta - n = 0.7 \geq 0. \quad (19)$$

With  $\theta = 135^\circ$ , it follows that  $|\sin 135^\circ - \cos 135^\circ| = \sqrt{2} = 1.4 + \beta_1$ , where  $\beta_1 > 0$ , and Eq. (15) becomes

$$x'_2 = x'_0 + 1 + \beta, \quad (20)$$

where  $\beta = \beta_1 + 0.4$  and  $\beta \geq 0.4$ . Substituting Eq. (16) into Eq. (20), the resulting  $x'_2$  can be written as

$$x'_2 = n + 1 + \alpha + \beta, \beta \geq 0.4, \alpha = 0.7. \quad (21)$$

Grouping terms in Eq. (21),  $x'_2$  takes the form

$$x'_2 = n + 2 + \varepsilon \quad (22)$$

$$0 < \varepsilon = \alpha + \beta - 1.0 < 1. \quad (23)$$

Before the rotated image point is drawn on the screen, the integer values of  $x'$  and  $y'$  in Eq. (1) must be determined, either by truncation or by rounding off. In the present theorem, the subscript character "t" in  $R_{135t}$ , means that truncation is used afterwards. Using truncation, the results for  $x'_0$  and  $x'_2$  are  $\text{TRUNCATE}(x'_0) = \text{TRUNCATE}(n + \alpha) = n$  and  $\text{TRUNCATE}(x'_2) = \text{TRUNCATE}(n + 2 + \varepsilon) = n + 2$ , respectively.

Since  $\alpha = 0.7$  from Eq. (18) and  $0 < \varepsilon < 1$  from Eqs. (22) and (23), both  $\alpha$  and  $\varepsilon$  are truncated. Note that, from Eq. (20), we have  $\beta \geq 0.4$  for the fixed angle  $\theta (= 135^\circ)$ . For Eq. (23) to hold, we must have  $\alpha \geq 1 - \beta = 0.6$  or

$\alpha \geq 0.6$ . Thus, if  $\alpha$  is chosen as 0.7, then the condition  $\alpha \geq 0.6$  will be satisfied. Q.E.D.

The above proof reveals that the difference between the integer values of  $x'_0$  and  $x'_2$  is always greater than or equal to 2. Alternatively, after rotation, the original horizontal line segment  $x'_0$  to  $x'_2$  of length 2 becomes a horizontal line segment  $x'_0$  to  $x'_2$  of length 3; i.e., there is a one point horizontal gap (a hole) between  $p'_0$  and  $p'_2$ .

**Case 2**  $x'_0 > x'_2$ . From Eq. (14), it follows that

$$x'_0 = x'_2 + |\sin\theta - \cos\theta| \text{ with } \theta = 135^\circ. \quad (24)$$

Comparing Eq. (24) with Eq. (15), we find that Eq. (24) is nothing but Eq. (15) with the roles of  $x'_0$  and  $x'_2$  interchanged. Therefore, the derivation steps for Case 2 should be exactly the same as those for Case 1 with  $x'_0$  and  $x'_2$  interchanged. The proof of Case 2 is thus omitted to save space. It follows that there also exists a horizontal hole between  $p'_0$  and  $p'_2$  in this case. Q.E.D.

The proof of condition (13) is as follows:

*Proof.* From Eq. (1), the following result can be obtained:

$$\begin{aligned} |x'_3 - x'_1| &= |x \cos\theta - (y+1)\sin\theta - (x+1)\cos\theta \\ &\quad + y \sin\theta| = |\sin\theta + \cos\theta|, \end{aligned} \quad (25a)$$

$$\begin{aligned} |y'_3 - y'_1| &= |x \sin\theta + (y+1)\cos\theta - (x+1)\sin\theta \\ &\quad - y \cos\theta| = |\sin\theta - \cos\theta|. \end{aligned} \quad (25b)$$

Taking  $\theta = 135^\circ$ , both  $|\sin\theta + \cos\theta| = 0$  and  $|\sin\theta - \cos\theta| > 1$ . Since  $|\sin\theta + \cos\theta| = 0$  in this case, the same reasoning used for (12) leads to consideration of Case 3:  $y'_3 > y'_1$  as before, where

$$y'_3 = y'_1 + |\sin\theta - \cos\theta| \text{ and } \theta = 135^\circ; \quad (26)$$

and Case 4:  $y'_3 < y'_1$  likewise, where

$$y'_1 = y'_3 + |\sin\theta - \cos\theta| \text{ and } \theta = 135^\circ. \quad (27)$$

Comparing the above Eq. (26) with the former Eq. (15), we find that Eq. (26) is simply a rewriting of Eq. (15) except that  $x'_0$  and  $x'_2$  in Eq. (15) are now replaced by  $y'_1$  and  $y'_3$  in Eq. (26), respectively. This is also true for Eqs. (27) and (24). It follows that proof of Cases 3 and 4 proceeds along the same lines as it does for Cases 1 and 2 above; thus, a (vertical) hole between  $p'_1$  and  $p'_3$  is seen to exist. Summarizing the proofs of Cases 1-4 above, it is

verified that a hole is generated for  $\theta = 135^\circ$ ; therefore rotation  $R_{135r}$  is not an I.R. Q.E.D.

Having proved Theorem 1, we can now prove the following.

**Theorem 2.** *The rotation  $R_{135r}$  is not an I.R.*

*Proof.* The derivation steps preceding the post-integer-conversion are almost exactly the same as Eq. (11) to Eq. (18) in Theorem 1, so most of the steps are omitted here. Also, only the post-integer-conversion of the present proof is considered. For reference, Eqs. (16) and (21) can be rewritten as Eqs. (28) and (29) as follows:

$$x'_0 = n + \alpha, \quad (28)$$

$$x'_2 = n + 1 + \alpha + \beta. \quad (29)$$

Next, post-integer-conversion is accomplished by means of the round-off function. Note that  $\theta = 135^\circ$  implies that  $0.414... < \beta < 0.5$ . Note also that the values of  $\alpha$  and  $\beta$  can be found via the round-off function, such that the new values of  $x'_0$  and  $x'_2$  become  $\text{ROUND}(x'_0) = n$ , and  $\text{ROUND}(x'_2) = n + 2$ , respectively. This is easy to show. Instead of choosing  $\alpha = 0.7$ , as in Theorem 1, the desired result can be obtained by letting  $\alpha = 0.4$ .

The same reasoning used in Theorem 1 shows that there are many sets  $(x, y, n)$  of values which validate Eq. (28). This proves that there is a (horizontal) hole between  $p'_0$  and  $p'_2$ . The same argument used in Theorem 1 can be applied to the cases of  $y'_1$  and  $y'_3$ . It, therefore, follows that  $R_{135r}$  is not an I.R. Q.E.D.

The proof of general rotation angles is presented in the following.

**Theorem 3.** *The rotation  $R_{\theta r}$  is not an I.R.*

In this theorem, both condition (12):  $\text{ASSOC}(p'_0, p'_2) = 1$  and condition (13):  $\text{ASSOC}(p'_1, p'_3) = 1$  have to be proved.

*Proof.* With respect to condition (12), the equations to be checked for  $\text{ASSOC}(p'_0, p'_2) = 1$  are as follows:

$$\begin{aligned} |x'_0 - x'_2| &= |x \cos\theta - y \sin\theta - (x+1)\cos\theta \\ &\quad + (y+1)\sin\theta| = |\sin\theta - \cos\theta|, \end{aligned} \quad (30)$$

$$\begin{aligned} |y'_0 - y'_2| &= |x \sin\theta + y \cos\theta - (x+1)\sin\theta \\ &\quad - (y+1)\cos\theta| = |\sin\theta + \cos\theta|. \end{aligned} \quad (31)$$

Note that, by definition,  $\text{ASSOC}(p'_0, p'_2) = 1$  implies that  $\sqrt{|x'_2 - x'_0|^2 + |y'_2 - y'_0|^2} > \sqrt{2}$ ; thus, either  $|x'_0 - x'_2|^2 > 2$  or  $|y'_2 - y'_0|^2 > 2$ . For the proof, only the larger of these terms is needed while the smaller one can be ignored. To justify the use of only this larger term, it is necessary to consider the following cases:

**Case 1.** When  $\theta$  (not a multiple of  $90^\circ$ ) is in the second or the fourth quadrant,  $|\sin\theta - \cos\theta| > |\sin\theta + \cos\theta|$ , and the larger term is  $|x'_0 - x'_2|$ . From Eq. (30), if  $x'_2 \geq x'_0$ , it follows that

$$x'_2 = x'_0 + |\sin\theta - \cos\theta| \quad (32)$$

while if  $x'_0 \geq x'_2$ , it is true that

$$x'_0 = x'_2 + |\sin\theta - \cos\theta|. \quad (33)$$

When  $\theta_{\min} = 90^\circ$  and  $\theta_{\max} = 180^\circ$ , or when  $\theta_{\min} = 270^\circ$  and  $\theta_{\max} = 360^\circ$ , for all  $\theta$  such that  $\theta_{\min} < \theta < \theta_{\max}$  ( $\theta$  is not a multiple of  $90^\circ$ ),

$$\begin{aligned} & |\sin\theta - \cos\theta| \\ &= \left| \sqrt{(\sin\theta - \cos\theta)^2} \right| = \left| \sqrt{\cos^2\theta + \sin^2\theta - 2\sin\theta\cos\theta} \right| \\ &= \left| \sqrt{1 - \sin 2\theta} \right| = 1 + \beta, \text{ and } 0 < \beta, \end{aligned}$$

$$\text{where } 90^\circ < \theta < 180^\circ \text{ or } 270^\circ < \theta < 360^\circ. \quad (34)$$

Since  $\sin 2\theta < 0$ , it follows that  $1 - \sin 2\theta > 1$  and, therefore, that  $\left| \sqrt{1 - \sin 2\theta} \right| = \sqrt{1 - \sin 2\theta} > 1$  while  $|\sin\theta - \cos\theta|$  can be set to  $1 + \beta$ , with  $\beta > 0$ . Therefore, Eq. (32) can be written as

$$x'_2 = x'_0 + 1 + \beta, \quad \text{where } 0 \leq \beta. \quad (35)$$

Since  $x'_0$  is a floating-point number,  $x'_0$  can be expressed as

$$x'_0 = n + \alpha, \quad n \in \mathbb{Z}, 0 \leq \alpha < 1. \quad (36)$$

Substituting Eq. (36) into Eq. (35),  $x'_2$  can be written as

$$x'_2 = x'_0 + 1 + \beta = n + \alpha + 1 + \beta$$

or

$$x'_2 = n + 1 + \alpha + \beta. \quad (37)$$

For the truncation case, the condition  $\alpha + \beta \geq 1$  is required in Eq. (37). Since  $0 \leq \alpha < 1$  and  $\beta > 0$ , this truncation requirement can certainly be met. In contrast, the round-

off case requires that  $\alpha < 0.5$  in Eq. (36), and that  $\alpha + \beta \geq 0.5$  in Eq. (37). This round-off requirement can similarly be met. For example, in Theorems 1 and 2 above,  $\theta = 135^\circ$ ,  $\beta = 0.414\dots$  and  $\beta = 0.4$  can, therefore, certainly be chosen. Hence, with  $\alpha$  and  $\beta$  both chosen as 0.4, it follows that  $\alpha + \beta = 0.4 + 0.4 = 0.8 \geq 0.5$ .

In fact, from elementary trigonometry, the values of  $\beta$  in Eq. (34) vary from  $\beta_{\min} = 0$  to  $\beta_{\max} = \sqrt{2} - 1$  (or 0.414...). In this range, one can choose values of  $\alpha$  and  $\beta$  which satisfy the requirements of both the truncation and round-off functions. For Eq. (33), with the roles  $x'_0$  and  $x'_2$  interchanged, exactly the same reasoning can be applied. The implication from this is that

$$\text{TRUNCATE}(x'_0) = n, \text{ and } \text{TRUNCATE}(x'_2) = n + 2 \quad (38)$$

and

$$\text{ROUND}(x'_0) = n, \text{ and } \text{ROUND}(x'_2) = n + 2. \quad (39)$$

That is, there is a hole between  $p'_0$  and  $p'_2$ . Having discussed the  $x$  coordinates of  $p'_0$  and  $p'_2$ , our focus can be placed on their corresponding  $y$  coordinates (Case 2).

**Case 2.** In Eq. (31), when  $\theta$  (not a multiple of  $90^\circ$ ) is in the first or the third quadrant,  $|\sin\theta + \cos\theta| > |\sin\theta - \cos\theta|$ , and  $|y'_0 - y'_2|$  is the larger term. From Eq. (31), it follows that

$$\begin{aligned} |\sin\theta + \cos\theta| &= \sqrt{(\sin\theta + \cos\theta)^2} = \sqrt{\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta} \\ &= \sqrt{1 + \sin 2\theta} = 1 + \beta, \text{ and } 0 < \beta < 1, \end{aligned}$$

$$\text{where } 0^\circ < \theta < 90^\circ \text{ or } 180^\circ < \theta < 270^\circ. \quad (40)$$

In this case, with  $\sin 2\theta > 0$ , Eq. (40) is found to hold. Using the same approach as in Case 1, it can be shown that there also exists a hole between  $p'_0$  and  $p'_2$ . Q.E.D.

The proof of (13)  $\text{ASSOC}(p'_1, p'_3) = 1$  is as follows:

*Proof.* In this case, the related equations are

$$\begin{aligned} |x'_1 - x'_3| &= |x \cos\theta - (y+1)\sin\theta - (x+1)\cos\theta + y \sin\theta| \\ &= |\sin\theta + \cos\theta|, \end{aligned} \quad (41)$$

$$\begin{aligned} |y'_1 - y'_3| &= |x \sin\theta + (y+1)\cos\theta - (x+1)\sin\theta - y \cos\theta| \\ &= |\sin\theta - \cos\theta|. \end{aligned} \quad (42)$$

Comparing Eq. (41) and Eq. (42) with Eq. (30) and Eq.

(31), it is seen that Eq. (41) and Eq. (42) are exactly the same as Eq. (30) and Eq. (31) except that  $|x'_0 - x'_2|$  (Eq. (30)) and  $|y'_0 - y'_2|$  (Eq. (31)) are replaced by  $|y'_1 - y'_3|$  (Eq. (42)) and  $|x'_1 - x'_3|$  (Eq. (41)), respectively.

Hence, the derivation steps for the present condition (13) should be the same as those for condition (12) and can thus be omitted. Following the same argument used in condition (12), we have proved condition (13), that a hole is generated between  $p'_1$  and  $p'_3$ . Q.E.D.

In the above discussion, one thing is worth mentioning: For the inequality  $\sqrt{|x'_1 - x'_3|^2 + |y'_1 - y'_3|^2} > 2$  to hold, when  $\theta$  is in the first or third quadrant, the larger term is  $|x'_1 - x'_3|^2$ . When  $\theta$  is in the second or fourth quadrant, the larger term is, instead,  $|y'_1 - y'_3|^2$ .

Combining (12) and (13) above, it follows that for all quadrants, there are cases in which there is a hole (both horizontally and vertically) surrounded by the four rotated points  $p'_0, p'_1, p'_2$  and  $p'_3$ , regardless of whether the truncation or the round-off conversion is applied afterwards. That is, neither  $R_{\theta_t}$  nor  $R_{\theta_r}$  is an I.R.s.

Having discussed the conditions under which holes may occur, it may be useful to discuss some hole-free cases.

Let  $p_0, p_1, p_2$  and  $p_3$  form a 4NAP in the plane P. After either an  $R_{sr}$  or  $R_{st}$  rotation, the corresponding rotated points  $p'_0, p'_1, p'_2$  and  $p'_3$  are calculated as in Eqs. (3)–(6). Assume that  $\theta$ , the angle of rotation, is such that  $\theta = n \times 90^\circ, n \in I$ . There are four cases to be considered.

**Case 1.** When  $n = 4k, k \in I, \cos\theta = 1$ , and  $\sin\theta = 0$ , Eqs. (3) – (6) can be simplified to

$$\begin{aligned} p'_0 &= (x, y), p'_1 = (x + 1, y), p'_2 = (x + 1, y + 1), \\ \text{and } p'_3 &= (x, y + 1). \end{aligned} \quad (43)$$

**Case 2.** When  $n = 4k + 1, k \in I, \cos\theta = 0$ , and  $\sin\theta = 1$ , Eqs. (3) – (6) can be simplified to

$$\begin{aligned} p'_0 &= (-y, x), p'_1 = (-y, x + 1), p'_2 = (-(y + 1), x + 1), \\ \text{and } p'_3 &= (-y, x). \end{aligned} \quad (44)$$

**Case 3.** When  $n = 4k + 2, k \in I, \cos\theta = -1$ , and  $\sin\theta = 0$ , Eqs. (3) – (6) can be simplified to

$$\begin{aligned} p'_0 &= (-x, -y), p'_1 = (-(x + 1), -y), \\ p'_2 &= (-(x + 1), -(y + 1)), \text{ and } p'_3 = (-x, -(y + 1)). \end{aligned} \quad (45)$$

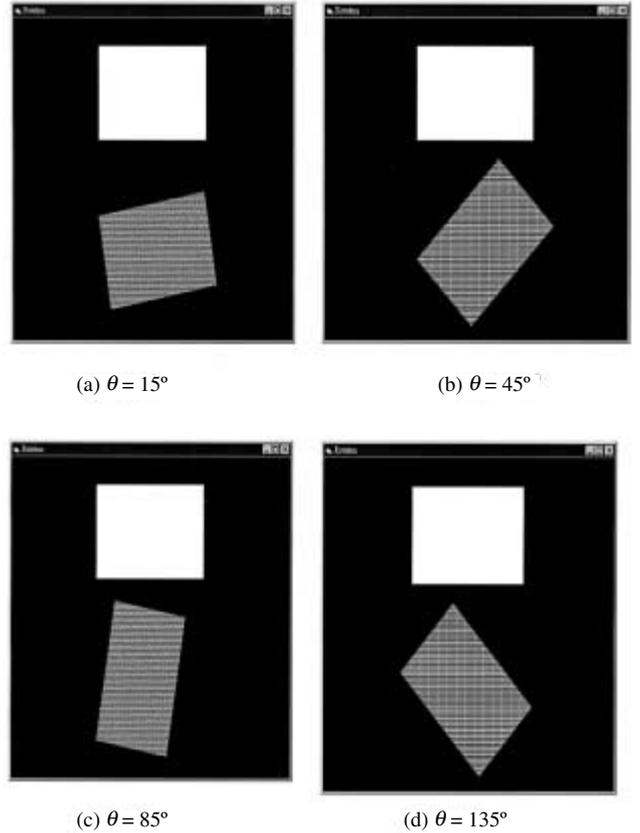
**Case 4.** When  $n = 4k + 3, k \in I, \cos\theta = 0$ , and  $\sin\theta = -1$ , Eqs. (3) – (6) can be simplified to

$$\begin{aligned} p'_0 &= (y, -x), p'_1 = (y, -(x + 1)), p'_2 = (y + 1, -(x + 1)), \\ \text{and } p'_3 &= (y + 1, -x). \end{aligned} \quad (46)$$

Observing the above equations (Eqs. (43) – (46)) carefully, it is seen that  $\text{ASSOC}(p'_i, p'_j) = 0, \forall i, j, 0 \leq i \leq 3, 0 \leq j \leq 3$ , and  $i \neq j$  in all cases. From Definition 5,  $p'_0, p'_1, p'_2$  and  $p'_3$  form a 4NAP, which implies that there does not exist a hole surrounded by  $p'_0, p'_1, p'_2$  and  $p'_3$ . Note that it is not difficult to prove that the above conclusion is true, regardless of whether the truncation or the round-off function is applied later in the post-integer-conversion. Thus, both Theorem 5 and Theorem 6 are seen to hold, and both  $R_{sr}$  and  $R_{st}$  are I.R.s.

To enhance our understanding of the above theoretical derivations, experiments were conducted using Visual Basic as the implementation language in the experiments. Parts (a), (b), (c), and (d) in Fig. 4 accord with Theorems 1-4 while parts (a), (b), (c), and (d) in Fig. 5 accord with Theorems 5 and 6.

The experiments show clearly that every part of Fig. 4 has generated holes (in its rotated image) while no part of Fig. 5 has generated holes.



**Fig. 4.** An original rectangular image and its rotated images for various rotation angles.

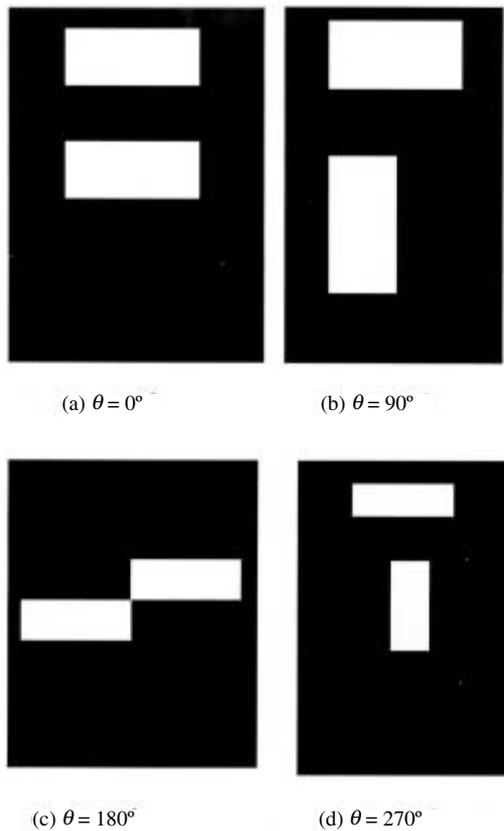


Fig. 5. An original rectangular image and its rotated images for special rotation angles.

## IV. Conclusions

This paper has proposed a mathematical model for the most frequently used image rotation formula, Eq. (1), to deal with the hole effect. This model includes, among other things, the associative function and post-integer-conversion functions. Using this model, it has been shown

that for some rotation angles, the rotated image may have points lacking relation to any point of the original image; i.e., there are holes in the rotated image.

In order to draw the rotated pixels, usually either the truncation or the round-off function (post-integer-conversion) is used to get the integer values of the coordinates for the floating-point values. In the past, many people believed that the hole effect was due to this treatment. However, this is not true. The truth is that long before post-integer-conversion was applied, hole effects had already been witnessed. Post-integer-conversion has absolutely no effect on whether or not a hole occurs. Rotating images without the occurrence of holes is clearly of both theoretical and practical value.

This paper has made some contributions in this regard. Firstly, the associative function, post-integer-conversion, 4NAP, 4FAP, and ideal rotation have been used together to construct a mathematical model. In addition, the various hole effects have classified into six theorems. Finally, experiments have been conducted under MS Windows 98 to illustrate the validity of the approach. This model can, therefore, be used to design new hole-free rotation algorithms.

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## 影像旋轉演算法問題之研究

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### 摘 要

在眾多傳統的教科書及許多研究者的論文之中常用的影像旋轉法，除了直角及其倍數角外，其他角度的旋轉，都常常會產生『洞』。許多影像演算法都指明有洞及如何避免『洞』，然而對『洞』的產生的原因機制則少有探討。本論文以八個定義及六個定理歸納影像旋轉的案例，從而嚴格的證明：不論用四捨五入法或截斷法（**truncation**）決定影像旋轉後之整數坐標值，都會天生自然的產生『洞』，『洞』多的時候影像會失真不好看。是以無洞的影像旋轉演算法，除了在學術探討方面，有其重要性及必要性外，在實用方面，亦深具其應用價值。我們的主要貢獻：一、運用關聯函數、四近關聯點、四遠關聯點及理想旋轉等建構一個數學模式；二、將洞效應歸納成六個定理，並予證明；三、以實驗驗證數學模式是可信的。因此，本論文對於構思新的無洞影像旋轉演算法會有助益。