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A Necessary and Sufficient Condition on Decomposable Convex Programming

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ABSTRACT

This paper considers a kind of decomposible convex programming: (P) $\min\{f(x);\ x\in C\}$, and its corresponding decomposible variational inequality $DVI(f, C)$, where $f(x):=f_1(x_1)+f_2(x_2)+\ldots+f_n(x_n)$, $\forall x:=(x_1, x_2, \ldots, x_n)$ and $C:=C_1\times C_2\times\cdots\times C_n$. Under the constraint qualification $0\in ri(\pi_{i=1}^{n}(coD(\partial f_i)-C_i))$, we show that x is a solution to $DVI(f, C)$ if, and only if, x is an optimal solution of (P).

Key Words: maximal monotone, (BH)-operator, Y^*-operator, duality mapping, decomposible variational inequality, decomposible convex programming

I. Introduction

Suppose that f_i is a proper closed convex function on a reflexive Banach space X_i, and that C_i is a nonempty closed convex subset of X_i, for each i=1, 2, 3, $\cdots$, n. Define

$$f(x):=f_1(x_1)+f_2(x_2)+\cdots+f_n(x_n),\ \forall x:=(x_1, x_2, \ldots, x_n),$$

$$X:=X_1\times X_2\times\cdots\times X_n,$$

and

$$C:=C_1\times C_2\times\cdots\times C_n.$$

In this paper, we consider the decomposible convex programming

(P) $\min\{f(x);\ x\in C\}$,

and the corresponding decomposible variational inequality

DVI(f, C): Find $x:=(x_1, x_2, \cdots, x_n)\in C$ and $x^*:=(x_1^*, x_2^*, \cdots, x_n^*)\in \partial f(x)$ such that

$$\langle y_i-x_i, x_i^*\rangle\geq 0,\quad \forall y_i\in C_i,\ \forall i=1, 2, \cdots, n,$$

where ∂f denotes the subdifferential operator of f, defined by

$$\partial f(x):=\{x^*\in X^*;\ f(z)-f(x)\geq\langle z-x, x^*\rangle,\ \forall z\in X\}.$$

For the convex function

$$f(x):=f_1(x_1)+f_2(x_2)+\cdots+f_n(x_n),\ \forall x:=(x_1, x_2, \cdots, x_n),$$

we have

$$\partial f(x)=\partial f_1(x_1)\times\partial f_2(x_2)\times\cdots\times\partial f_n(x_n).$$

Also, we define the normality operator of the set C to be

$$N_C(x):=\{x^*\in X^*;\ \langle y-x, x^*\rangle\leq 0,\ \forall y\in C\},\ \forall x\in C.$$

Indeed, the normality operator of C is just the subdifferential of the indicator function δ_C, where

$$\delta_C(x):=\begin{cases} 0, & \forall x\in C \\ +\infty, & \forall x\notin C \end{cases}.$$

Thus, we have (see also Clarke, 1989)

$$N_C(x)=N_{C_1}(x_1)\times N_{C_2}(x_2)\times\cdots\times N_{C_n}(x_n),$$

$$\forall x:=(x_1, x_2, \cdots, x_n)\in C.$$

It can be shown that solving the problem $DVI(f, C)$ is equivalent to solving the usual variational inequality:

VI(f, C): Find $x\in C$ and $x^*\in\partial f(x)$ such that

$$\langle y-x, x^*\rangle\geq 0,\ \forall y\in C.$$

Equivalently, $x \in C$ satisfies the nonlinear equation

$$0 \in \partial f(x) + N_C(x).$$

In this paper, we shall only be concerned with the case where each X_i is a reflexive Banach space. Asplund (1967a), together with a result of Troyanski (1971), has shown by means of a theorem of Lindenstrauss that there exists an equivalent norm on X_i which is everywhere Fréchet differentiable except at the origin and whose polar norm on its dual X_i^* is everywhere Fréchet differentiable except at the origin. For simplicity of notation, we may assume that the given norm on each X_i already has these special properties. For a set-valued operator $T:X \to 2^{X^*}$, the domain, the range, the graph, and the inverse of T are denoted by

$$D(T) := \{x \in X;\ T(x) \neq \emptyset\},$$

$$R(T) := \cup\{T(x);\ x \in D(T)\},$$

$$G(T) := \{(x,\ x^*) \in X \times X^*;\ x^* \in T(x)\},$$

and

$$T^{-1}(x^*) := \{x \in X;\ x^* \in T(x)\}.$$

Recall that a set-valued operator $T:X \to 2^{X^*}$ is monotone if

$$\langle x' - x, y' - y \rangle \geq 0,\ \forall (x,\ y),\ (x',\ y') \in G(T).$$

T is maximal monotone if T is monotone and there exists no other monotone set-valued operator whose graph properly contains the graph of T. It is known (Rockafellar, 1966, 1970a, 1970b) that the subdifferential operator of a proper closed convex function is maximal monotone. T is called a (BH)-operator if

$$\inf_{(x, x^*) \in G(T)} \langle x - \overline{x}, x^* - \overline{x}^* \rangle > -\infty,$$

$$\forall \overline{x} \in D(T),\ \forall \overline{x}^* \in R(T).$$

For $Y^* \subset X^*$, T is called an Y^*-operator if, for all $\overline{y}^* \in Y^*$, there is some $\overline{x} \in X$ such that

$$\inf_{(x, x^*) \in G(T)} \langle x - \overline{x}, x^* - \overline{y}^* \rangle > -\infty.$$

Such operators have been studied extensively in both theory and applications. For details, see the work by Brézis (1973) and Phelps (1989) and the references cited therein. Also, we let J be the duality map (see Asplund, 1967b; Brézis, 1973; Cudia, 1974; Isac, 1992; Moreau, 1965, 1967; Rockafellar, 1970c; Troyanski, 1971). That is, $J:X \to X^*$ is a norm preserving map such that

$$\langle x, J(x) \rangle = \|x\|^2 = \|J(x)\|^2.$$

It is easy to show that

$$\frac{1}{2}\|z\|^2 \geq \frac{1}{2}\|x\|^2 + \langle z - x, J(x) \rangle, \quad \forall z \in X,$$

It follows that $J(x) \in \partial j(x)$, where $j(x) := \frac{1}{2}\|x\|^2$.

We remark that x is an optimal solution of (P) if, and only if, $0 \in \partial(f + \delta_C)(x)$, and in general we have

$$\partial f(x) + N_C(x) = \partial f(x) + \partial\delta_C(x) \subset \partial(f + \delta_C)(x). \qquad \textbf{(1)}$$

Therefore, the solution of $VI(f, C)$ is always an optimal solution of the problem (P) but not converse in general. In finite-dimensional space under the welll-known constraint qualification

$$ri(domf) \cap riC \neq \emptyset,$$

Rockafellar(1970a, theorem 23.8) shows that

$$\partial(f + \delta_C)(x) = \partial f(x) + N_C(x).$$

Under this constraint qualification, any optimal solution of (P) is also a solution to $VI(f, C)$. Here, riC denotes the relative interior of C; that is, the interior taken in the closed affine hull of C and $domf$ denote the effective domain of f, defined by

$$domf := \{x \in X;\ f(x) < +\infty\}.$$

In this paper we will show that x is a solution to $DVI(f, C)$ if, and only if, x is an optimal solution of (P), under the more general constraint qualification

$$0 \in ri(\pi_{i=1}^{n}(coD(\partial f_i) - C_i)).$$

It should be noted that our constraint qualification is definitely weaker than that of Rockafellar. A simple example in an infinite-dimensional Hilbert space X would be the following: suppose that y is a nonzero linear functional on X which is not continuous. Let V be a hyperplane in X (i.e., a linear variety of co-dimension 1) given by

$$V = \{x \in X;\ \langle x, y \rangle = 0\},$$

and let C be a one-dimensional subspace such that $X = V + C$. Let f be the indicator function of V. Then,

$domf=V=D(\partial f)$ is non-closed but convex and dense in X since y is not continuous in X. Thus,

$$ri(domf)=riV=int_{cl(affV)}V=int_X V=intV=\emptyset.$$

It follows that $0\in X=riX=ri(V+C)=ri(coD(\partial f)-C)$, but $ri(domf)\cap riC=\emptyset\cap riC=\emptyset$. Here, we denote by coA, $affA$, clA and $intA$ the convex hull, the affine hull, the closure, and the interior of A, respectively.

The following basic identity and inclusion (see Clarke, 1989; Rockafellar, 1970a) will be used later. For any closed convex function f on X, we have

$$ri(domf)=co(ri(domf))\subset coD(\partial f)\subset co(domf)=domf.$$

For any subsets A and B of X, we have

$$coA\times coB=co(A\times B),$$

and

$$coA+coB=co(A+B).$$

For any subsets A_i and B_i of X, $i = 1, 2, \cdots, n$, we have

$$(A_1\times A_2\times\cdots\times A_n)-(B_1\times B_2\times\cdots\times B_n)$$

$$=(A_1-B_1)\times(A_2-B_2)\times\ldots\times(A_n-B_n).$$

II. Preliminary Results

We begin with some well-known results, which we shall use in proving our main results.

Proposition II.1. (Rockafellar, 1970c) If $T:X\to 2^{X^*}$ is a monotone operator, then T is maximal monotone if, and only if, $R(T+J)=X^*$.

The following proposition, essential due to Browder (1968), is a generalization of the fundamental Hilbert space theorem of Minty (1961):

Proposition II.2. If $T:X\to 2^{X^*}$ is a maximal monotone operator and $\lambda>0$, then $R(T+\lambda J)=X^*$ and $(T+\lambda J)^{-1}$ is a single-valued maximal monotone operator from X^* to X, which is demicontinuous.

Proposition II.3. (Rockafellar, 1970c) If T_1, $T_2:X\to 2^{X^*}$ are maximal monotone operators such that

$$D(T_1)\cap intD(T_2)\neq\emptyset,$$

then T_1+T_2 is a maximal monotone operator.

Next, we will show a basic property. From this, we can conclude that the sets $ri(domf)$, $coD(\partial f)$, and $domf$ have the same closed affine hull.

Proposition II.4. Suppose that Y^* is a convex subset of X^* and that

$$\emptyset\neq riY^*\subset S\subset clY^*\subset X^*.$$

Then the sets S and Y^* have the same closed affine hull.

Proof. Let v be a common vector of S and riY^*, and let $A(S)$ and $V(S)$ denote the closed affine hull of S and the closed subspace of X^* generated by S, respectively. Then

$$A(S)=cl\ aff(S)=cl(v+span(S-S))$$

$$=v+cl\ span(S-S)=v+V(S).$$

Similarly, we have

$$A(Y^*)=cl\ aff(Y^*)=v+V(Y^*).$$

Thus, to show that $A(S)=A(Y^*)$, we only need to show that $V(S)=V(Y^*)$. It, therefore, suffices just to show

$$V(clY^*)\subset V(riY^*).$$

For any $u\in clY^*$, there exists a net (u_δ) in Y^* such that $u=\lim u_\delta$, and for $\lambda\in(0, 1]$ we define

$$u_\delta(\lambda):=(1-\lambda)u_\delta+\lambda\ v.$$

It is easy to check that $u_\delta(\lambda)\in riY^*$, and for each δ we have

$$\lim_{\lambda\downarrow 0} u_\delta(\lambda)=u_\delta.$$

This information implies that

$$u_\delta-v=\lim_{\lambda\downarrow 0}(u_\delta(\lambda)-v)\ \in cl\ span(riY^*-riY^*)$$

$$=V(riY^*).$$

It follows that

$$u=v+\lim(u_\delta-v)\in v+cl\ V(riY^*)=v+V(riY^*).$$

Hence, $clY^*\subset v+V(riY^*)$. It follows that

$$clY^*-clY^*$$

$$=(clY^*-v)-(clY^*-v)\subset V(riY^*)-V(riY^*)$$

$$=V(riY^*),$$

which implies that

$$V(clY^*)=cl\ span(clY^*-clY^*)\subset cl\ span V(riY^*)$$
$$=V(riY^*).$$

Therefore, we can conclude that riY^* and clY^* have the same closed affine hull, and the assertion follows.

Using the above propositions, we now establish a technical result, which is the tool used to prove our main theorems.

Proposition II.5. If $T:X\to 2^{X^*}$ is a maximal monotone Y^*-operator and $R(T)\subset cl(coY^*)$, then $ri(coY^*)\subset R(T)$. Moreover, if $ri(coY^*)\neq\emptyset$, then $ri(coY^*)=riR(T)$.

Proof. We first show that T is a coY^*-operator. Let $\overline{y}^*=\sum_i \lambda_i \overline{y}_i^*$, where $\overline{y}_i^*\in Y^*$, $\lambda_i\geq 0$, $\sum\lambda_i=1$, $\forall i=1, 2, \cdots$. Since T is an Y^*-operator, for each i there are $\overline{x}_i\in X$ and $\mu_i>-\infty$ such that

$$\mu_i\leq\langle x-\overline{x}_i, x^*-\overline{y}_i^*\rangle,\quad \forall(x, x^*)\in G(T).$$

Equivalently,

$$\mu_i-\langle\overline{x}_i, \overline{y}_i^*\rangle\leq\langle x, x^*\rangle-\langle x, \overline{y}_i^*\rangle-\langle\overline{x}_i, x^*\rangle,$$
$$\forall(x, x^*)\in G(T).$$

Let $\overline{x}:=\sum_i\lambda_i\overline{x}_i$. Then, for $(x, x^*)\in G(T)$ we have

$$\sum_i\lambda_i\mu_i-\sum_i\lambda_i\langle\overline{x}_i, \overline{y}_i^*\rangle$$
$$\leq\langle x, x^*\rangle-\left\langle x, \sum_i\lambda_i\overline{y}_i^*\right\rangle-\left\langle\sum_i\lambda_i\overline{x}_i, x^*\right\rangle.$$

Define

$$\mu:=\sum_i\lambda_i\mu_i-\sum_i\lambda_i\langle\overline{x}_i, \overline{y}_i^*\rangle.$$

It follows that

$$-\infty<\mu+\langle\overline{x}, \overline{y}^*\rangle$$
$$\leq\langle x, x^*\rangle-\langle x, \overline{y}^*\rangle-\langle\overline{x}, x^*\rangle+\langle\overline{x}, \overline{y}^*\rangle$$
$$=\langle x-\overline{x}, x^*-\overline{y}^*\rangle.$$

We can, therefore, conclude that T is a coY^*-operator. So, now we may suppose without loss of generality that Y^* is convex. Note that, for any $y^*\in riY^*$, there is some $\alpha>0$, so that whenever $z^*\in V:=cl\ span(Y^*-Y^*)$ with $\|z^*\|\leq\alpha$, we have $y^*+z^*\in Y^*$. Since T is an Y^*-operator, there exist some $\overline{x}(z^*)\in X$ and $\mu(z^*)>-\infty$ such that

$$\mu(z^*)\leq\langle x-\overline{x}(z^*), x^*-y^*-z^*\rangle,\ \forall(x, x^*)\in G(T). \tag{2}$$

By Proposition II.2, for $\epsilon>0$ there is some $u_\epsilon\in X$ such that

$$y^*\in(T+\epsilon J)(u_\epsilon), \tag{3}$$

which implies that

$$(u_\epsilon, y^*-\epsilon Ju_\epsilon)\in G(T). \tag{4}$$

Combining Eq. (2) with Eq. (4), we then have

$$\mu(z^*)\leq\langle u_\epsilon-\overline{x}(z^*), -\epsilon Ju_\epsilon-z^*\rangle \tag{5}$$

Since $J(\bullet)\in\partial(\frac{1}{2}\|\bullet\|^2)$,

$$\langle\overline{x}(z^*)-u_\epsilon, \epsilon J u_\epsilon\rangle\leq\frac{\epsilon}{2}\|\overline{x}(z^*)\|^2-\frac{\epsilon}{2}\|u_\epsilon\|^2.$$

It follows that if $z^*\in V$ with $\|z^*\|\leq\alpha$, and $\epsilon>0$, then

$$\langle u_\epsilon, z^*\rangle\leq\langle u_\epsilon, z^*\rangle+\frac{\epsilon}{2}\|u_\epsilon\|^2$$
$$\leq\frac{\epsilon}{2}\|\overline{x}(z^*)\|^2+\langle\overline{x}(z^*), z^*\rangle-\mu(z^*). \tag{6}$$

Using the above result, we now prove that, for all $z^*\in V$,

$$\sup_{0<\epsilon\leq 1}|\langle u_\epsilon, z^*\rangle|<+\infty. \tag{7}$$

Define

$$\beta(z^*):=\frac{1}{2}\|\overline{x}(z^*)\|^2+\langle\overline{x}(z^*), z^*\rangle-\mu(z^*),$$

and

$$\gamma(z^*):=\max\{|\beta(z^*)|, |\beta(-z^*)|\}.$$

By Eq. (6), for $z^*\in V$ with $\|z^*\|\leq\alpha$, and $0<\epsilon\leq 1$, we have

$$|\langle u_\epsilon, z^*\rangle|\leq\gamma(z^*). \tag{8}$$

For $z^*\in V$ with $\|z^*\|>\alpha$, we let $\lambda:=\frac{\|z^*\|}{\alpha}$ and $z_1^*:=\frac{1}{\lambda}z^*$; then $z^*=\lambda z_1^*$ and $\|z_1^*\|=\alpha$. It follows from Eq. (8) that

$$|\langle u_\epsilon, z_1^*\rangle|\leq\gamma(z_1^*),\quad\forall 0<\epsilon\leq 1.$$

Hence, we have

$$|\langle u_\epsilon, z^*\rangle|=\lambda|\langle u_\epsilon, z_1^*\rangle|\leq\lambda\gamma(z_1^*)=\frac{\|z^*\|}{\alpha}\gamma(\frac{\alpha}{\|z^*\|}z^*),$$
$$\forall 0<\epsilon\leq 1. \tag{9}$$

Combining Eq. (8) and Eq. (9) yields

$$\sup_{0<\epsilon\leq 1} |\langle u_\epsilon, z^*\rangle| < +\infty, \quad \forall z^* \in V.$$

Next, we define ${}^{\perp}V := \{x \in X \mid \langle x, y^*\rangle = 0, \ \forall y^* \in V\}$ and let $U := X/{}^{\perp}V$. Since X is assumed to be reflexive, we may identify the dual space U^* (see Rudin (1991, theorem 4.9)) with $({}^{\perp}V)^{\perp}$, which is V. Thus, for $x \in X$, we may define $[x] := x + {}^{\perp}V \in U$ and define $\langle [u], \bullet\rangle : U^* \to R$ by $\langle [u], u^*\rangle := \langle u, u^*\rangle$. It is easy to check that the map is well-defined since for all $y \in {}^{\perp}V$ and $u^* \in U^* = V$, we have

$$\langle [u], u^*\rangle = \langle u + y, u^*\rangle = \langle u, u^*\rangle$$

By Eq. (7), we then have

$$\sup_{0<\epsilon\leq 1} |\langle [u_\epsilon], u^*\rangle| < +\infty, \ \forall u^* \in U^*.$$

From the uniform boundedness principle, we obtain

$$\sup_{0<\epsilon\leq 1} \|[u_\epsilon]\| < +\infty. \tag{10}$$

Now we will show that, for all $u \in X$ with $Ju \in V$,

$$\|u\| \leq \|[u]\|. \tag{11}$$

For any $\delta > 0$, since

$$\|[u]\| := \inf\{\|u - y\| \mid y \in {}^{\perp}V\},$$

there exists some $y \in {}^{\perp}V$ such that

$$\|u - y\| < \delta + \|[u]\|.$$

It follows that, for any $Ju \in V$, we have $\langle y, Ju\rangle = 0$; therefore,

$$\|u\|^2 = \langle u, Ju\rangle = \langle u - y, Ju\rangle \leq \|u - y\| \|Ju\|$$

$$\leq (\delta + \|[u]\|)\|u\|.$$

Equivalently, $\|u\| \leq \delta + \|[u]\|$, $\forall \delta > 0$. Thus, for all $u \in X$ with $Ju \in V$, we have $\|u\| \leq \|[u]\|$. Note that, from Eq. (4), we see that

$$Ju_\epsilon \in \epsilon^{-1}(y^* - Tu_\epsilon) \subset \epsilon^{-1}(Y^* - R(T))$$

$$\subset \epsilon^{-1}(Y^* - clY^*) \subset cl\ span(Y^* - Y^*) = V. \tag{12}$$

By Eqs. (10), (11), and (12), we conclude that the set $\{u_\epsilon \mid 0 < \epsilon \leq 1\}$ is bounded. Since

$$\|u_\epsilon\|^2 = \langle u_\epsilon, Ju_\epsilon\rangle = \|Ju_\epsilon\|^2,$$

the set $\{Ju_\epsilon \mid 0 < \epsilon \leq 1\}$ is also bounded. So, when ϵ converges to 0, we may assume that $y^* - \epsilon Ju_\epsilon$ converges to y^*, and that u_ϵ converges weakly to some $u \in X$. Since T is monotone, by Eq. (4) we have

$$\langle z - u_\epsilon, z^* - (y^* - \epsilon Ju_\epsilon)\rangle, \ \forall (z, z^*) \in G(T).$$

Letting $\epsilon \to 0$, we obtain

$$0 \leq \langle z - u, z^* - y^*\rangle, \ \forall (z, z^*) \in G(T).$$

Since T is a maximal monotone operator, $(u, y^*) \in G(T)$; that is, $y^* \in R(T)$. Thus, $ri(coY^*) \subset R(T)$.

Moreover, we know that

$$\emptyset \neq ri(coY^*) \subset R(T) \subset cl(coY^*).$$

Applying Proposition II.4 with $S := R(T)$, we obtain that the sets $R(T)$ and coY^* have the same closed affine hall; call it A. Working in A, we take the interior operation int_A and get

$$int_A(coY^*) = int_A(ri(coY^*)) \subset int_A(RT) \subset int_A(cl(coY^*))$$

$$= int_A(coY^*).$$

It follows that

$$ri(coY^*) \subset riR(T) \subset ri(coY^*).$$

Thus, we conclude that $ri(coY^*) = riR(T)$.

III. Main Results

In this section, we establish the main results. Indeed, under the constraint qualification

$$0 \in ri(\pi_{i=1}^n (coD(\partial f_i) - C_i)),$$

the two problems $DVI(f, C)$ and (P) have the same solution set. Besides previous propositions in Section II, we need the following:

Lemma III.1. If $T: X \to 2^{X^*}$ is a monotone operator and $\lambda > 0$, then $T + \lambda J$ is a (BH)-operator.

Proof. Let $\overline{x} \in D(T + \lambda J) = D(T)$ and $\hat{x}^* \in R(T + \lambda J)$. For any $(x, x^*) \in G(T + \lambda J)$, there is some $y^* \in T(x)$ such that $x^* = y^* + \lambda Jx$. It follows that

$$\frac{\langle x - \overline{x}, x^*\rangle}{\|x\|} = \frac{\langle x - \overline{x}, y^*\rangle}{\|x\|} + \frac{\lambda\langle x - \overline{x}, Jx\rangle}{\|x\|}.$$

Notice that

$$\frac{\langle x-\overline{x}\,,y^*\rangle}{\|x\|}\geq-\|y^*\|-\frac{\|\overline{x}\|\|y^*\|}{\|x\|}.$$

Since $\frac{\|\overline{x}\|^2}{2}\geq\frac{\|x\|^2}{2}+\langle\overline{x}-x\,,Jx\rangle$, we have

$$\frac{\langle x-\overline{x}\,,Jx\rangle}{\|x\|}\geq\frac{\|x\|}{2}-\frac{\|\overline{x}\|^2}{2\|x\|}.$$

It follows that

$$\frac{\lambda\langle x-\overline{x}\,,Jx\rangle}{\|x\|}\geq\lambda\left(\frac{\|x\|}{2}-\frac{\|\overline{x}\|^2}{2\|x\|}\right);$$

hence,

$$\frac{\langle x-\overline{x}\,,x^*\rangle}{\|x\|}\geq-\|y^*\|-\frac{\|\overline{x}\|\|y^*\|}{\|x\|}+\lambda\left(\frac{\|x\|}{2}-\frac{\|\overline{x}\|^2}{2\|x\|}\right).$$

Thus, we can conclude that

$$\lim_{\substack{(x,x^*)\in G(T+\lambda J)\\ \|x\|\to+\infty}}\frac{\langle x-\overline{x}\,,x^*\rangle}{\|x\|}=+\infty.$$

Now, let $\alpha>0$ be such that for any $(x, x^*)\in G(T+\lambda J)$ with $\|x\|\geq\alpha$, we have

$$\frac{\langle x-\overline{x}\,,x^*\rangle}{\|x\|}\geq 1+\|\hat{x}^*\|.$$

If $\|x\|\geq\beta:=\|\overline{x}\|\|\hat{x}^*\|$, then

$$\frac{\langle x-\overline{x}\,,\hat{x}^*\rangle}{\|x\|}\leq\frac{\|x\|\|\hat{x}^*\|}{\|x\|}+\frac{\|\overline{x}\|\|\hat{x}^*\|}{\|x\|}\leq\|\hat{x}^*\|+1.$$

It follows that for $(x,x^*)\in G(T+\lambda J)$ with $\|x\|\geq\gamma:=\max\{\alpha,\beta\}$, we have

$$\frac{\langle x-\overline{x}\,,x^*\rangle}{\|x\|}\geq 1+\|\hat{x}^*\|\geq\frac{\langle x-\overline{x}\,,\hat{x}^*\rangle}{\|x\|},$$

which yields

$$\langle x-\overline{x}\,,x^*-\hat{x}^*\rangle\geq 0. \tag{13}$$

On the other hand, by the monotonicity of $T+\lambda J$, for $(x, x^*)\in G(T+\lambda J)$ with $\|x\|\leq\gamma$, and $(\overline{x}\,,\overline{x}^*)\in G(T+\lambda J)$, we have

$$\langle x-\overline{x}\,,x^*-\overline{x}^*\rangle\geq 0.$$

It follows that

$$\langle x-\overline{x},x^*-\hat{x}^*\rangle\geq\langle x-\overline{x},\overline{x}^*-\hat{x}^*\rangle\geq-\|x-\overline{x}\|\|\overline{x}^*-\hat{x}^*\|$$
$$\geq-(\gamma+\|\overline{x}\|)\|\overline{x}^*-\hat{x}^*\|.$$

Combining Eq. (13) with Eq. (14), we obtain

$$\inf_{(x,x^*)\in G(T+\lambda J)}\langle x-\overline{x}\,,x^*-\hat{x}^*\rangle$$
$$\geq\min\{0,-(\gamma+\|\overline{x}\|)\|\overline{x}^*-\hat{x}^*\|\}>-\infty.$$

This implies that $T+\lambda J$ is a (BH)-operator.

We next prove an extensive result of Brézis and Haraux (1976) in a reflexive Banach space. Indeed, Brézis and Haraux (1976, theorem 3) show that if T_1 and T_2 are monotone (BH)-operators from a Hilbert space into itself such that T_1+T_2 is maximal monotone, then $R(T_1+T_2)\cong R(T_1)+R(T_2)$; that is,

$$clR(T_1+T_2)=cl(R(T_1)+R(T_2)),$$

and

$$intR(T_1+T_2)=int(R(T_1)+R(T_2)).$$

Theorem III.2. If T_1, T_2:$X\to 2^{X^*}$ are monotone (BH)-operators such that T_1+T_2 is maximal monotone, then

$$ri(coR(T_1)+coR(T_2))\subset R(T_1+T_2)\subset R(T_1)+R(T_2).$$

Moreover, if $ri(coR(T_1)+coR(T_2))\neq\emptyset$, then

$$riR(T_1+T_2)=ri(R(T_1)+R(T_2))=ri(coR(T_1)+coR(T_2)).$$

Proof. Let T:=T_1+T_2, and Y^*:=$R(T_1)+R(T_2)$. Then $R(T)=R(T_1+T_2)\subset R(T_1)+R(T_2)=Y^*\subset cl(coY^*)$. For $\hat{y}^*=\hat{y}_1^*+\hat{y}_2^*$, where $\hat{y}_i^*\in R(T_i)$, $\forall i$=1, 2, and $\overline{x}\in D(T)$= $D(T_1)\cap D(T_2)$, since each T_i is a (BH)-operator, we have

$$-\infty<\inf_{(x,y_i^*)\in G(T_i)}\langle x-\overline{x}\,,y_i^*-\hat{y}_i^*\rangle=:\mu_i.$$

Thus, we have

$$-\infty<\mu_1+\mu_2$$
$$\leq\inf_{(x,y_i^*)\in G(T_i)}\langle x-\overline{x}\,,(y_1^*+y_2^*)-(\hat{y}_1^*+\hat{y}_2^*)\rangle$$
$$=\inf_{(x,x^*)\in G(T)}\langle x-\overline{x}\,,x^*-\hat{y}^*\rangle.$$

It follows that T is an Y^*-operator. Notice that

$$R(T_1+T_2)\subset coR(T_1)+coR(T_2)$$

$$=co(R(T_1)+R(T_2))\subset cl(co(R(T_1)+R(T_2))).$$

Now by Proposition II.5 with $T=T_1+T_2$ and $Y^*=R(T_1)+R(T_2)$, we have

$$ri(coR(T_1)+coR(T_2))=ri(co(R(T_1)+R(T_2)))$$

$$=riR(T_1+T_2)\subset R(T_1+T_2)\subset R(T_1)+R(T_2)\subset coR(T_1)+coR(T_2)$$

$$=co(R(T_1)+R(T_2))\subset cl(co(R(T_1)+R(T_2))).$$

Applying Proposition II.4, together with nonemptiness of $ri(coR(T_1)+coR(T_2))$, we conclude that the above sets have the same closed affine hull. Taking the interiors with respect to this common closed affine hull yields

$$riR(T_1+T_2)=ri(R(T_1)+R(T_2))$$

$$=ri(coR(T_1)+coR(T_2)).$$

The following theorem generalizes Rockafellar theorem 23.8 in Rockafellar (1970a) to a reflexive Banach space.

Theorem III.3. Under the constraint qualification

$$0\in ri\,(\pi_{i=1}^n\,(coD(\,\partial f_i\,)-C_i\,))\,,$$

the operator $(\partial f_1+N_{C_1})\times(\partial f_2+N_{C_2})\times\ldots\times(\partial f_n+N_{C_n})$ is maximal monotone. Moreover, one has

$$\partial(f+\delta_C)(x)=\partial f(x)+N_C(x).$$

Proof. We first show that

$$(\partial f+N_C)(x)=(\partial f_1+N_{C_1})(x_1)\times(\partial f_2+N_{C_2})(x_2)$$

$$\times\ldots\times(\partial f_n+N_{C_n})(x_n)$$

is a maximal monotone operator. By Proposition II.1, it is sufficient to prove just that

$$R(\partial f+N_C+J)=X^*.$$

For any fixed $x_0^*\in X^*$, we define $S_1,\ S_2{:}X^*\to X$ by

$$S_1(x^*):=(\,\partial f+\tfrac{1}{2}J\,)^{-1}(x^*)\,,$$

and

$$S_2(x^*):=-(\,N_C+\tfrac{1}{2}J\,)^{-1}(\,x_0^*-x^*\,)\,.$$

It is clear from Proposition II.2 that S_1 and S_2 are maximal monotone single-valued operators. Thus,

$$D(S_1)\cap intD(S_2)=x^*\neq\emptyset.$$

By Proposition II.3, we conclude that S_1+S_2 is also a maximal monotone operator. Since by Lemma III.1 each S_i^{-1} is a (BH)-operator, each S_i is also a (BH)-operator. It follows from Theorem III.2 that

$$ri(coR(S_1)+coR(S_2))\subset R(S_1+S_2).$$

Since

$$R(S_1)=D(\partial f)=D(\partial f_1)\times D(\partial f_2)\times\ldots\times D(\partial f_n),$$

and

$$R(S_2)=-D(N_C)=-C=-(C_1\times C_2\times\ldots\times C_n),$$

it follows that

$$0\in ri\,(\,\pi_{i=1}^n\,(\,coD(\,\partial f_i\,)-C_i\,)\,)$$

$$=ri\,(\,\pi_{i=1}^n\,coD(\,\partial f_i\,)-\pi_{i=1}^n C_i\,)$$

$$=ri\,(\,co\,(\,\pi_{i=1}^n\,D(\,\partial f_i\,)\,)-C\,)$$

$$=ri\,(\,coD(\,\partial f)-C\,)$$

$$=ri\,(\,coR\,(S_1)+coR\,(S_2)\,)\subset R\,(\,S_1+S_2\,)\,.$$

Thus, $0\in R(S_1+S_2)$. Let $x^*\in X^*$ be such that $0\in S_1(x^*)+S_2(x^*)$. It follows that there is some $y^*\in S_1(x^*)$ such that $-y^*\in S_2(x^*)$. We then have

$$x_0^*=x^*+(x_0^*-x^*)\in(\,\partial f+\tfrac{1}{2}J)\,(y^*)+(N_C+\tfrac{1}{2}J)\,(y^*)$$

$$=(\partial f+N_C+J\,)\,(y^*)\,.$$

We conclude that $\partial f+N_C$ is a maximal monotone operator. Since, by Eq. (1),

$$G(\partial f+N_C)\subset G(\partial(f+\delta_C)),$$

it follows that

$$G(\partial f+N_C)=G(\partial(f+\delta_C));$$

hence,

$$(\partial f+N_C)(x)=\partial(f+\delta_C)(x).$$

Finally, we are ready to prove the main theorem, which shows that x is a solution to $DVI(f, C)$ if, and only if, x is an optimal solution of (P) under a kind of constraint qualification.

Theorem III.4. Under the constraint qualification

$$0 \in ri\,(\pi_{i=1}^{n}\,(coD(\,\partial f_i) - C_i)),$$

the two problems $DVI(f,C)$ and (P) have the same solution set.

Proof. By the previous remark, it is well-known that every solution to $DVI(f,C)$ is an optimal solution of (P). Thus, it is sufficient to show that every optimal solution of (P) is also a solution to $DVI(f,C)$. Let x be an optimal solution of (P). Then we have

$$0 \in \partial(f+\delta_C)(x).$$

By Theorem III.3, we obtain

$$0 \in \partial(f+\delta_C)(x) = \partial f(x) + N_C(x).$$

It follows that x is also a solution to $DVI(f, C)$. Thus, we complete the proof.

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一個關於可分割凸規劃的充要條件

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摘　要

本文考慮以下的可分割凸規劃問題(P):$min\{f(x);\ x\in C\}$和對應的可分割變分不等式$DVI(f, C)$：找$x:=(x_1, x_2, \ldots, x_n)\in C$和$x^*:=(x_1^*, x_2^*, \ldots, x_n^*)\in \partial f(x)$使得$\langle y_i - x_i, x_i^*\rangle \geq 0$, $\forall\, y_i\in C_i, \forall i=1, 2, \ldots, n$，其中$f(x):=f_1(x_1)+f_2(x_2)+\ldots+f_n(x_n)$, $\forall x:=(x_1, x_2, \ldots, x_n)$，且$C:=C_1\times C_2\times\ldots\times C_n$。在下面的限制條件下$0\in ri\,(\pi_{i=1}^{n}\,(coD\,(\partial f_i)-C_i))$，我們證明了:$x$是問題$(P)$的一個最優解的充要條件為$x$也是問題$DVI(f, C)$的一個解。